

Gravitational Waves

Phenomenology and Tackling

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Primer

Einstein's Field Equation [8],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (1)$$

Weak-field linear approximation as a perturbation $h_{\mu\nu}$ about a flat background Minkowski metric $\eta_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (2)$$

Decomposing the spatial part and expanding in terms of the distance of observation [7],

$$h_{ij}^{\text{TT}} \simeq \frac{12G}{r c^4} \ddot{\mathcal{I}}_{ij} \left(t - \frac{r}{c} \right), \quad (3)$$

expressing leading *quadrupole* behaviour.

Overlapping Gravitational Waves

Time Domain overlap [6, 10, 11]

$$h = h_1 + h_2. \quad (4)$$

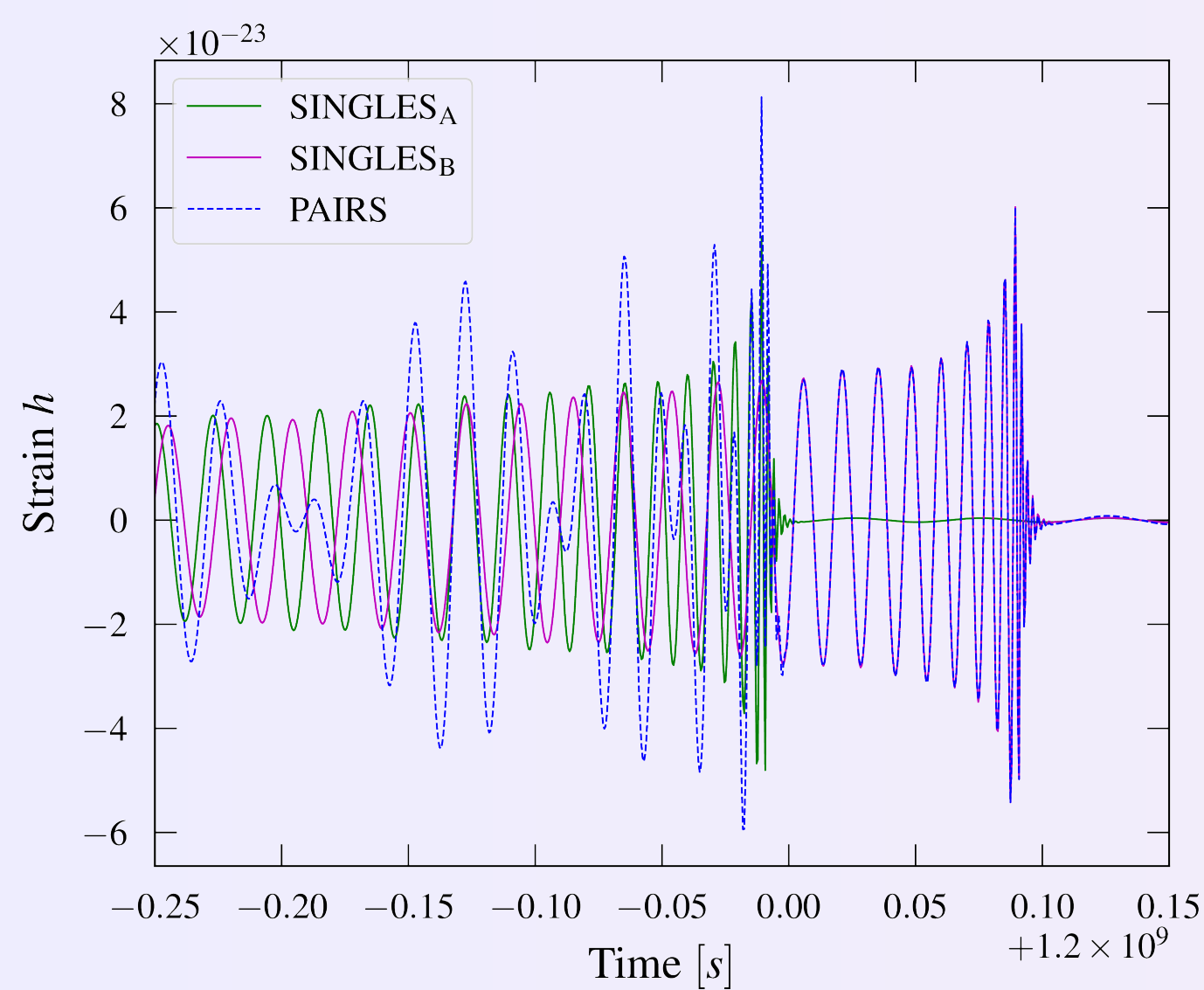


Figure 1: Illustration of two distinct gravitational wave signals overlapping in time.

Gravitational Lensing

Lensing in Waveforms

$$h_L(\Theta, \lambda_L) = h_U(\Theta) \mathcal{F}(f; \lambda_L). \quad (5)$$

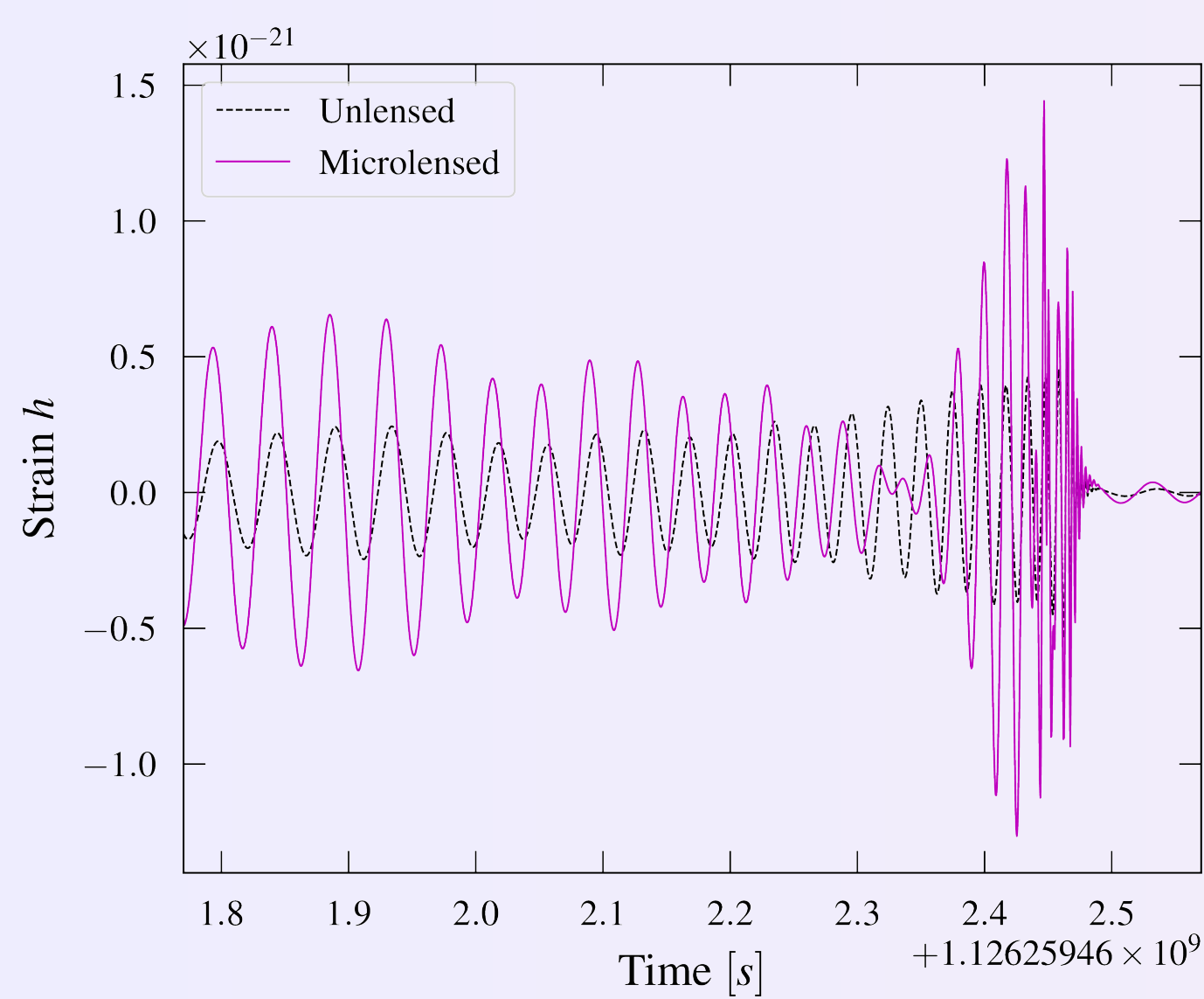


Figure 2: A comparison of an unlensed gravitational waveform with a microlensed counterpart.

Amplification Factor, for a lensed signal,

$$\mathcal{F}(f; \mathbf{y}) = -if / d^2 \mathbf{x} \exp \{ i 2\pi f \tau_D(\mathbf{x}, \mathbf{y}) \} \quad (6)$$

with $\tau_d(\mathbf{x}, \mathbf{y})$ representing the time delay function.

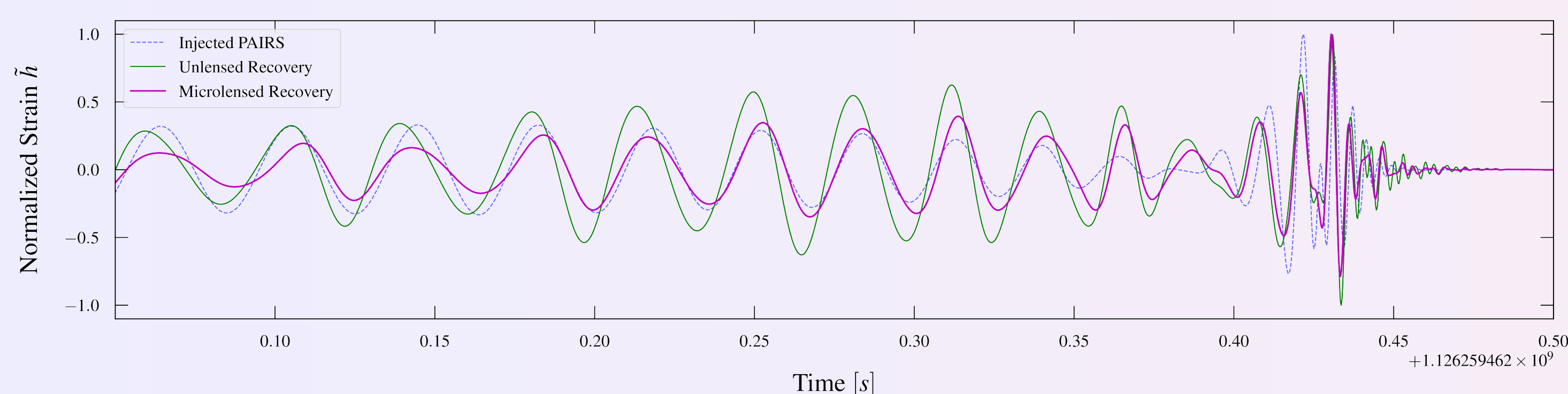


Figure 3: Demonstration of a microlensed waveform template being optimally fitted to a simulated overlapping signal.

Inference

For an isolated point-mass lens of redshifted mass M_{ℓ}^z , we define $\omega = \frac{8\pi G}{c^3} M_{\ell}^z$, and,

$$\mathcal{F}(f; \omega, y) = \exp \left\{ \frac{\pi\omega}{4} + i \frac{\omega}{2} \left[\ln \frac{\omega}{2} - 2\Phi_m(y) \right] \right\} \times \Gamma \left(1 - i \frac{\omega}{2} \right) {}_1F_1 \left(\frac{\omega}{2}, 1; i \frac{\omega y^2}{2} \right) \quad (7)$$

with $\Phi_m(y) = \frac{1}{2}(x_m - y)^2 - \ln x_m$, and $x_m = \frac{y}{2} + \frac{1}{2}\sqrt{y^2 + 4}$.

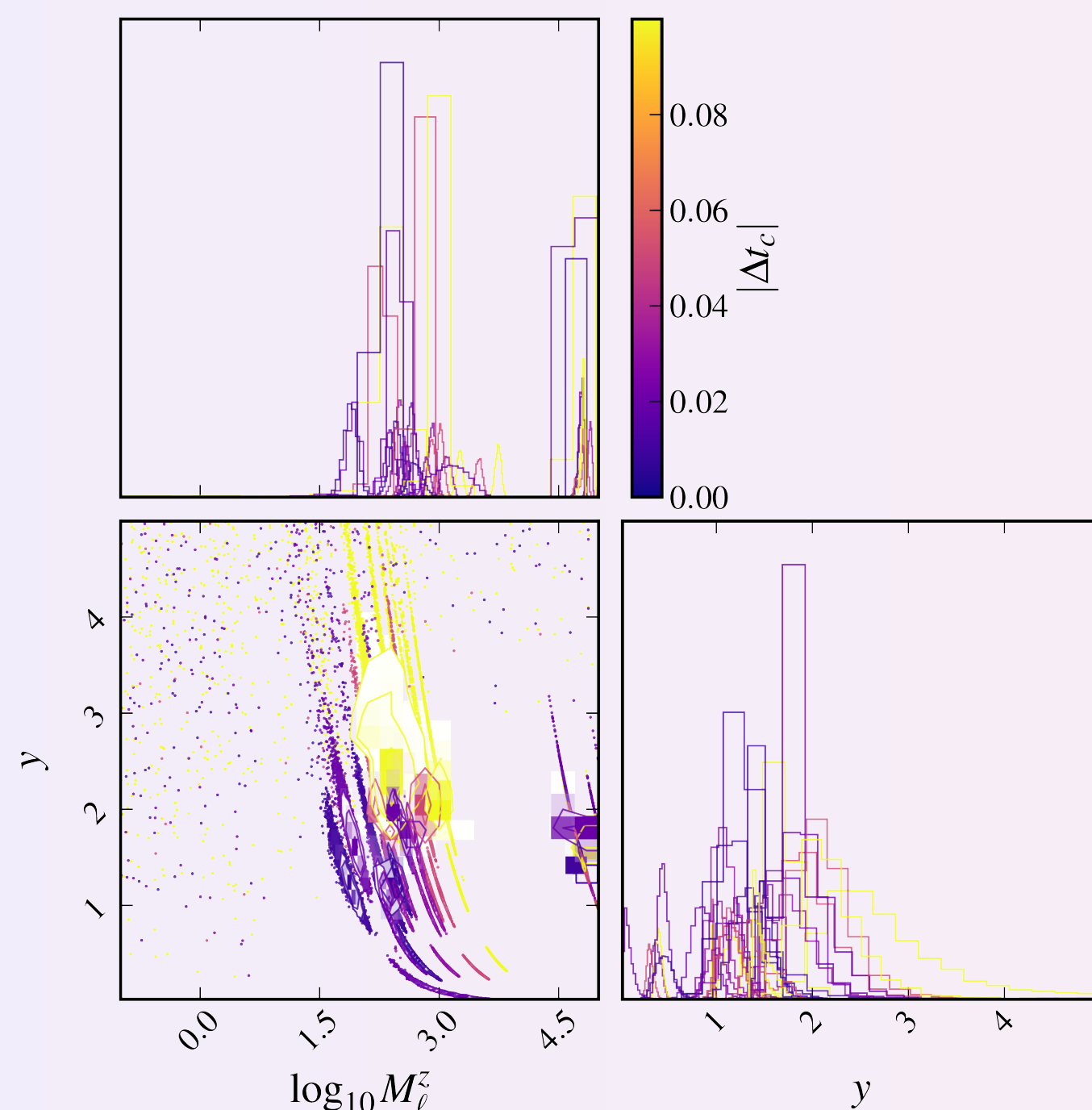


Figure 4: Posterior probability distributions for the gravitational lens parameters, M_{ℓ}^z and the impact parameter y from the parameter estimation of overlapping signals with microlensed signals.

Parameter Estimation

Residual noise, $\mathbf{n} = \mathbf{d} - \mathbf{h}$, [1]

$$p(\mathbf{n}) = \frac{1}{(\det(2\pi\mathbf{C}))^{\frac{N}{2}}} \exp \left(-\frac{1}{2} \chi^2 \right), \quad (8)$$

where $\chi^2 = \langle \mathbf{d} - \mathbf{h} | \mathbf{d} - \mathbf{h} \rangle$.

Bayesian Inference

$$p(\vec{\lambda} | \mathbf{d}) = \frac{\mathcal{L}(\mathbf{d} | \vec{\lambda}) \Pi(\vec{\lambda})}{\mathcal{Z}(\mathbf{d})}, \quad (9)$$

where, Likelihood for Stationary Gaussian noise, $\mathcal{L}(\mathbf{d} | \vec{\lambda}) = \exp \left(-\frac{1}{2} \langle \mathbf{d} - \mathbf{h} | \mathbf{d} - \mathbf{h} \rangle \right)$.

Noise weighted inner product, [7]

$$\langle \mathbf{a} | \mathbf{b} \rangle = \mathbf{a}^\dagger \cdot \mathbf{C}^{-1} \cdot \mathbf{b} = a_i \mathbf{C}_{ij}^{-1} b_j, \quad (10)$$

Frequency domain,

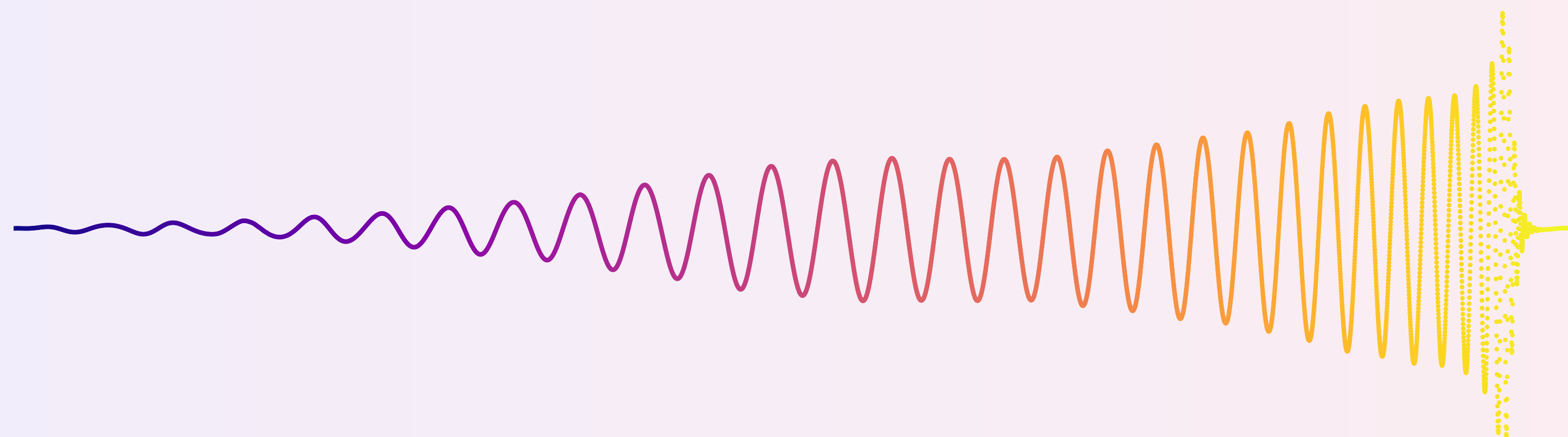
$$\langle \mathbf{a} | \mathbf{b} \rangle = 4\Re \left\{ \sum_f \frac{\tilde{\mathbf{a}}^*(f) \tilde{\mathbf{b}}(f)}{S_n(f)} \right\}, \quad (11)$$

Likelihood ratio, $\Lambda(\vec{\lambda})$,

$$\Lambda(\vec{\lambda}) = \frac{p(\mathbf{d} | \mathbf{h}(\vec{\lambda}), \mathcal{H}_1)}{p(\mathbf{d} | \mathcal{H}_0)}, \quad (12)$$

and for Gaussian noise, $\Lambda(\vec{\lambda}) = \exp \left\{ -\langle \mathbf{d} | \mathbf{h} \rangle + \frac{1}{2} \langle \mathbf{h} | \mathbf{h} \rangle \right\}$.

Maximised likelihood, over intrinsic parameters, further, w.r.t. overall amplitude, $\rho(\vec{\lambda}) = \langle \mathbf{d} | \mathbf{h}(\vec{\lambda}) \rangle$, with Signal-to-noise ratio (SNR), as $\text{SNR} = \sqrt{\langle \mathbf{h} | \mathbf{h} \rangle}$,



Inspiral-Merger-Ringdown Consistency Tests

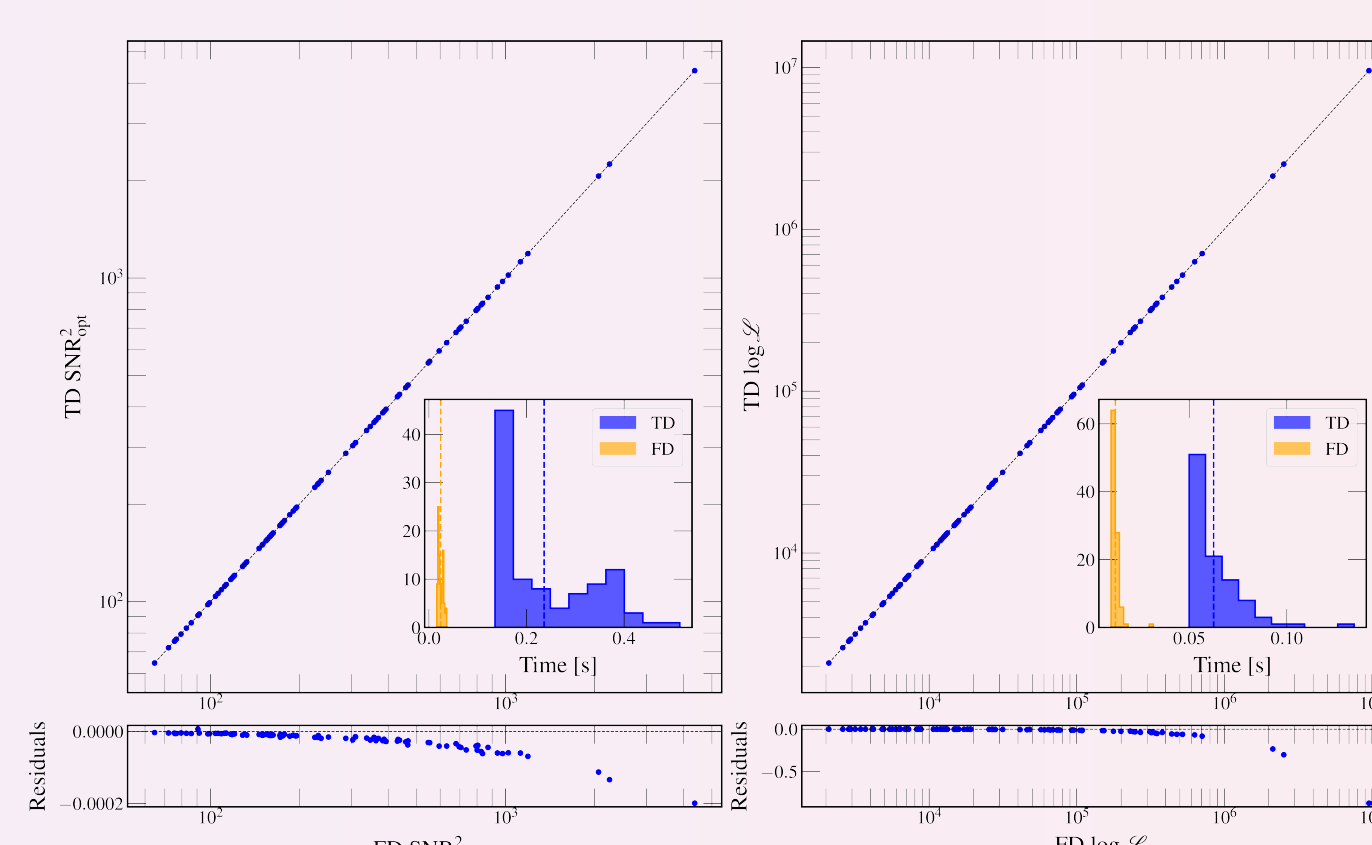


Figure 5: A direct comparison between frequency and time-domain methods for calculating the optimal SNR and Likelihood. The inset histograms show that the time-domain is computationally expensive.

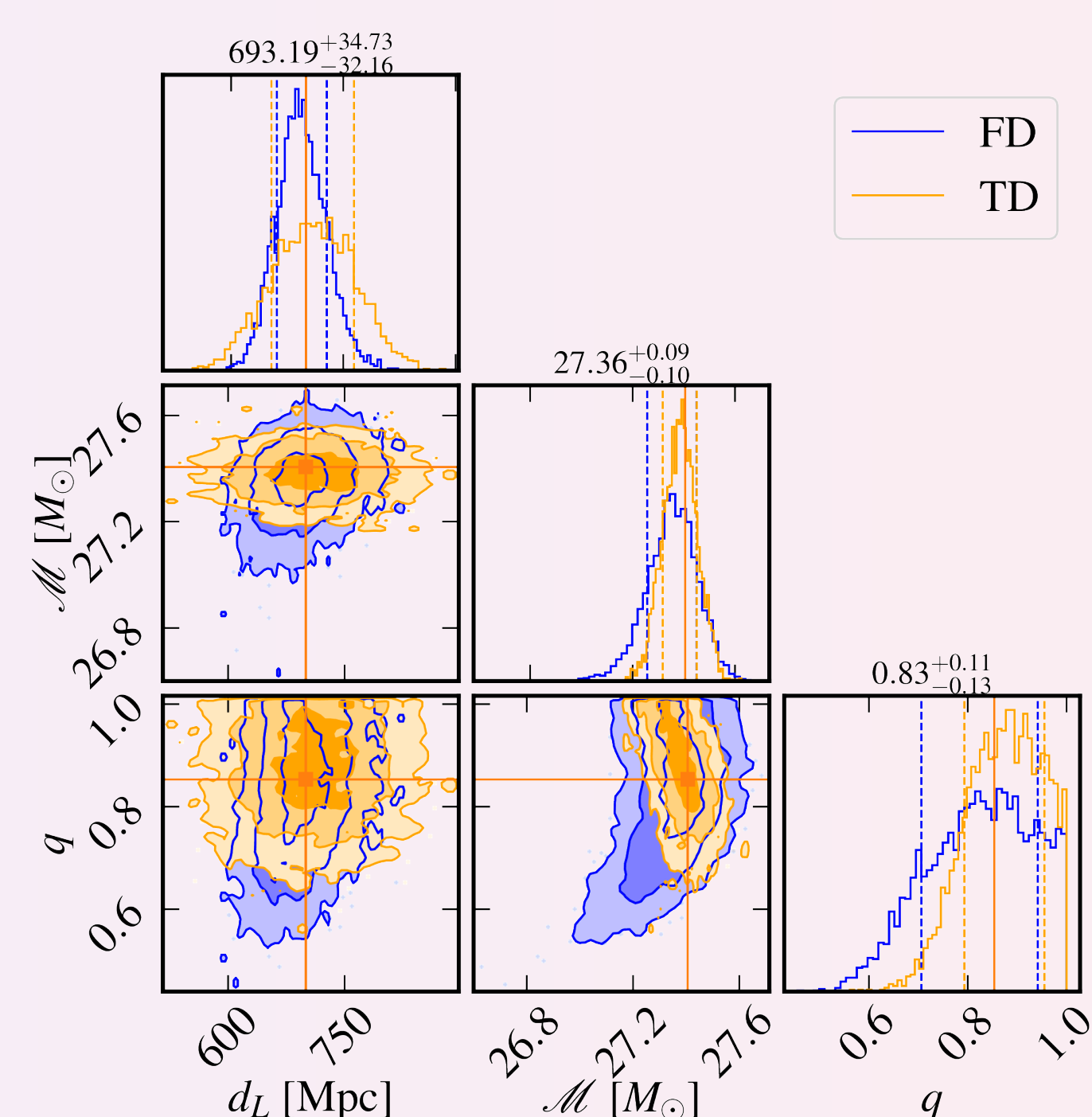


Figure 6: Corner plot comparing the posterior distributions for intrinsic source parameters estimated using frequency and time-domain analyses.

Quadratic Modes in Ringdown

Quadratic modes [4],

$$\omega_{lmn} = \omega_{\ell_1 m_1 n_1} + \omega_{\ell_2 m_2 n_2}, \quad (14)$$

$$\frac{1}{\tau_{lmn}} = \frac{1}{\tau_{\ell_1 m_1 n_1}} + \frac{1}{\tau_{\ell_2 m_2 n_2}}, \quad (15)$$

$$\mathcal{A}_{lmn} \propto \mathcal{A}_{\ell_1 m_1 n_1} \mathcal{A}_{\ell_2 m_2 n_2}, \quad (16)$$

$$\Phi_{lmn} = \Phi_{\ell_1 m_1 n_1} + \Phi_{\ell_2 m_2 n_2} + \text{const.} \quad (17)$$

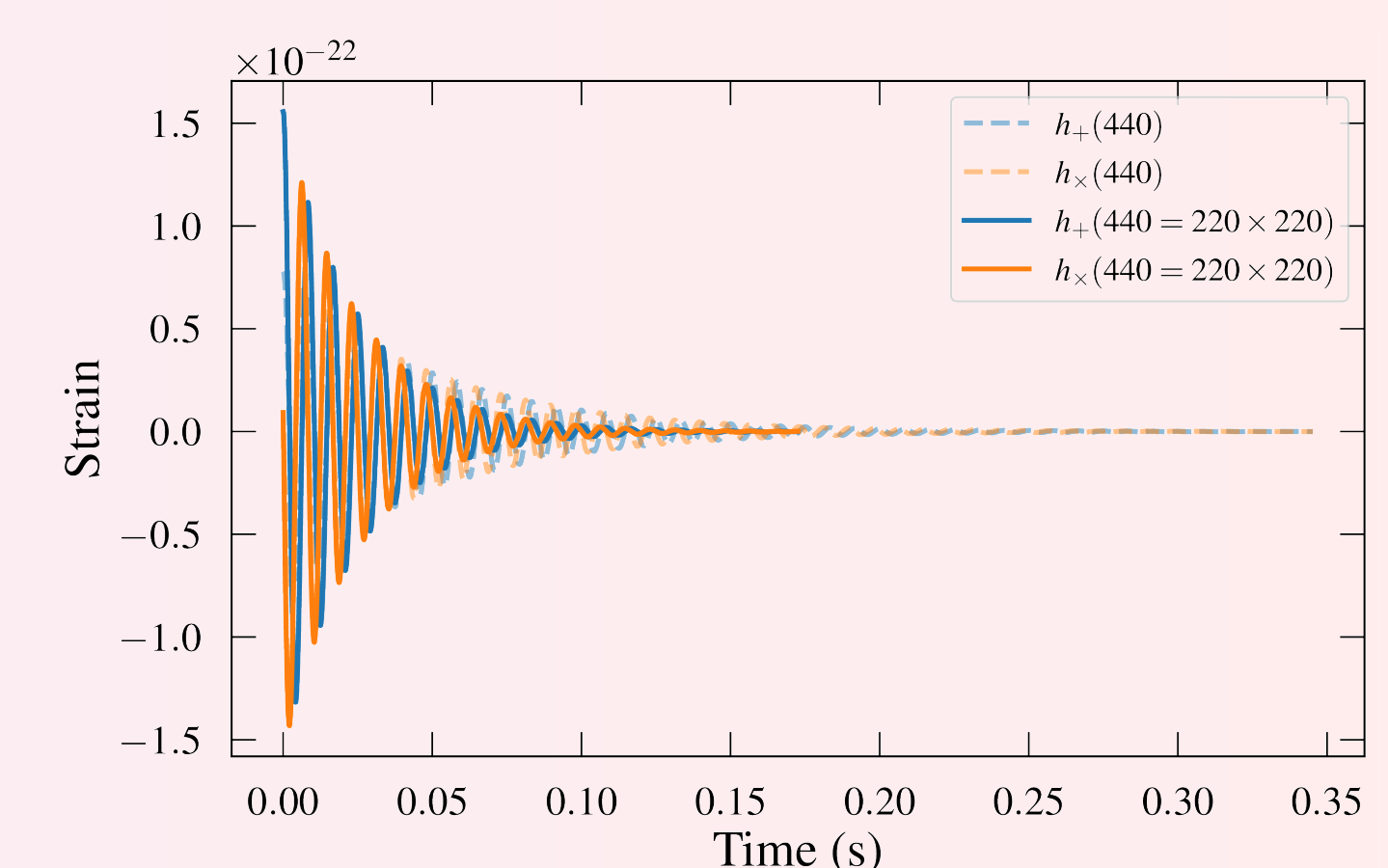


Figure 7: Representation of Linear and Quadratic Quasi-Normal Modes in Ringdown waveform.

Ringdown Amplitudes

Non-circular parameterization [2–4],

$$A_{\bar{h}}(\tau) = \left(\frac{c_1^A}{1 + \exp(-c_2^A \tau + c_3^A)} + c_4^A \right)^{\frac{1}{c_5^A}}, \quad (18)$$

$$\phi_{\bar{h}}(\tau) = -c_1^\phi \ln \left(\frac{1 + c_3^\phi \exp(-c_2^\phi \tau)}{1 + c_3^\phi} \right) + \frac{c_4^\phi \exp(-2c_2^\phi \tau)}{1 + c_3^\phi + c_4^\phi}. \quad (19)$$

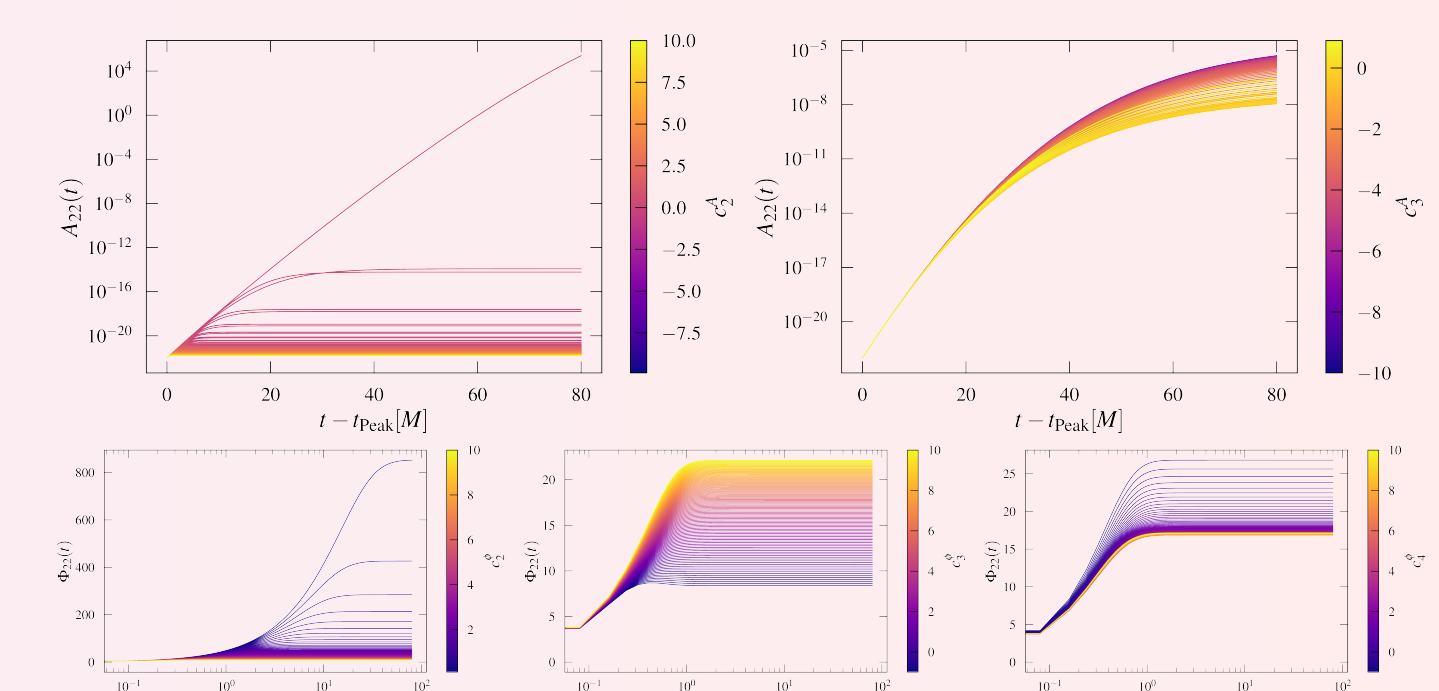


Figure 8: Parametric Fits for the Amplitude and Phase of the Ringdown of the Waveform.

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