

L01.01. Cosmography using Standard Sirens

Basics of Cosmology

Cosmological Principle (Large-Scale):

- **Homogeneity:** Invariance under translations
Fall in variance in matter overdensity when smoothened on radius R
- **Isotropy:** Invariance under rotations
Constant temperature of CMBR in any direction
- **Time-Asymmetry:** Evolution through time

Fundamental Observers: $R_{\text{phys}} = a(t)R_{\text{com}}$, comoving coordinates R_{com} for initial observers at $t = t_0$, with the physical distance R_{phys} , scaling by a scale factor $a(t)$.

Metric

Metric obeying *Homogeneity & Isotropy*:

$$ds^2 = -c^2 dt^2 + a^2(t) d\Sigma^2$$

with time set to proper time for fundamental observers. Isotropy implies conformal $a(t)$ for the spatial metric $d\Sigma^2$. Foliate spacetime into time and homogeneous and isotropic slices Σ_t .

Embedding d dimension *maximally symmetric surface* in $d + 1$ dimension with a constraint

equation. $d\Sigma^2 = \sum_{i=1}^d dx_i^2 + \kappa^2 d\alpha^2$, where we transform for 3D, as $\alpha = R \cos\left(\frac{\chi}{R}\right)$,

$x = R \cos \theta \sin\left(\frac{\chi}{R}\right)$, $y = R \sin \theta \sin \phi \sin\left(\frac{\chi}{R}\right)$, $z = R \sin \theta \cos \phi \sin\left(\frac{\chi}{R}\right)$, to get

$$d\Sigma^2 = \begin{cases} d\chi^2 + \chi^2 (d\theta^2 + \sin^2 \theta d\phi^2) & \text{Flat Space} \\ d\chi^2 + R^2 \sin^2\left(\frac{\chi}{R}\right) (d\theta^2 + \sin^2 \theta d\phi^2) & \text{Positive Curvature} \\ d\chi^2 + R^2 \sinh^2\left(\frac{\chi}{R}\right) (d\theta^2 + \sin^2 \theta d\phi^2) & \text{Negative Curvature} \end{cases}$$

with generalization

$$d\Sigma^2 = d\chi^2 + f^2(\chi) d\Omega^2$$

with

$$f(\chi) = \begin{cases} \chi & \text{Flat Space} \\ \sin\left(\frac{\chi}{R}\right) & \text{Positive Curvature} \\ \sinh\left(\frac{\chi}{R}\right) & \text{Negative Curvature} \end{cases}$$

Standard form of the **FLRW** metric,

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

Dynamics

Velocities $v = \frac{dR_{\text{phys}}}{dt} = \frac{\dot{a}}{a} R_{\text{phys}} + a(t) \frac{dR_{\text{com}}}{dt}$, with *cosmological recession velocity*: $\frac{\dot{a}}{a} R_{\text{phys}}$, *peculiar velocity*: $a(t) \frac{dR_{\text{com}}}{dt}$ which is short scales.

Hubble Law, $v = H_0 d$, overestimate of H_0 , expanding of the universe since $\dot{a} > 0$.

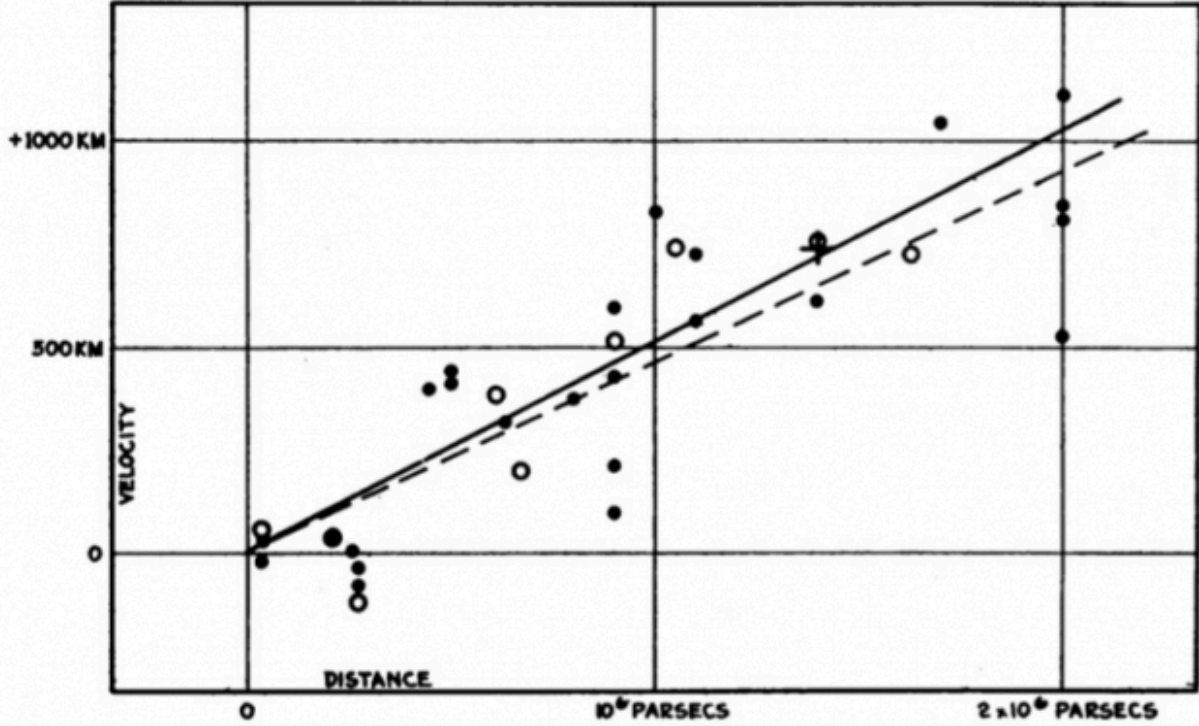


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Redshift, $ds^2 = 0$ for photons in an expanding universe, $c\delta t = a(t)\delta\chi$, thereby, $\int_{t_e}^{t_r} \frac{c\delta t}{a(t)} = \int \delta\chi$ to give the total distance traveled from emission at $t = t_e$ to reception at $t = t_r$. For photon wavelength of $c\delta t$, we have $\int_{t_e+\delta t_e}^{t_r+\delta t_r} \frac{c\delta t}{a(t)} = \int \delta\chi$, thereby equating $\int_{t_e}^{t_r} \frac{c\delta t}{a(t)} = \int_{t_e+\delta t_e}^{t_r+\delta t_r} \frac{c\delta t}{a(t)}$, we have $\frac{c\delta t_e}{a(t_e)} - \frac{c\delta t_r}{a(t_r)} = 0$, thereby, the wavelengths,

$$\frac{\lambda_r}{\lambda_e} = 1 + z = \frac{a(t_r)}{a(t_e)}$$

defined through the redshift z , ignoring peculiar velocities.

Distances

Angular distance

Relation between angles (*solid angle*) and physical size (*physical area*), related as

$$D_{\text{ang}} = \sqrt{\frac{A}{\Omega}} = \sqrt{\frac{4\pi f^2(\chi) a^2(t)}{4\pi}} = \frac{f(\chi)}{1+z}$$

where the area is of the 3-surface at constant time.

Luminosity Distance

Relation between luminosity of an object and the flux received by it,

Let us now consider the flux received from a distant source in the Universe, which has an intrinsic bolometric luminosity L^{bol} integrated over all wavelengths, and per unit wavelength, we have $dL^{\text{bol}} = L\mathcal{I}(\lambda)d\lambda$, where the intensity, averages as $\int \mathcal{I}(\lambda)d\lambda = 1$.

Evaluating the number of photons emitted per unit wavelength λ_e , in a time δt_e at time of emission t_e , given as

$$d\mathcal{N}(\lambda_e) = \frac{dL\delta t_e}{hc/\lambda_e} = L \frac{\lambda_e}{hc} \mathcal{I}(\lambda_e) d\lambda_e \delta t_e$$

For the light emitted by a sphere at coordinate χ , after time t_r , the surface area is $4\pi a^2(t_r) f^2(\chi)$, now at time t_r , where we chose $a(t_r) = 1$, and have the number of photons received per unit wavelength λ_r , over the entire surface of the sphere, given as

$$d\mathcal{N}(\lambda_r) = d\mathcal{N}_e \left(\frac{\lambda_r}{1+z} \right) = L \frac{\lambda_r}{(1+z)hc} \mathcal{I} \left(\frac{\lambda_r}{1+z} \right) \frac{d\lambda_r}{1+z} \delta t_e$$

since the wavelength changes as time proceeds.

Thus the amount of energy that passes through the sphere in a unit time interval is given by

$$\frac{d\mathcal{N}_r(\lambda_r)}{\delta t_r} \frac{hc}{\lambda_r} = L \frac{1}{(1+z)} \mathcal{I} \left(\frac{\lambda_r}{1+z} \right) \frac{d\lambda_r}{1+z} \frac{\delta t_e}{\delta t_r}$$

The flux $\mathcal{F}(\lambda_r)d\lambda_r$ is the amount of energy received per unit time per unit area perpendicular to the line of sight to the source. Thus $\mathcal{F}(\lambda_r)d\lambda_r$ should be equal to

$$\mathcal{F}(\lambda_r)d\lambda_r = \frac{d\mathcal{N}_r(\lambda_r)}{\delta t_r} \frac{hc}{\lambda_r} \frac{1}{4\pi f^2(\chi) a^2(t_r)}$$

which can be simplified as

$$\begin{aligned}
\mathcal{F}(\lambda_r)d\lambda_r &= \frac{d\mathfrak{N}_r(\lambda_r)}{\delta t_r} \frac{hc}{\lambda_r} \frac{1}{4\pi f^2(\chi)a^2(t_r)} \\
&= \frac{L}{(1+z)} \mathcal{I}\left(\frac{\lambda_r}{1+z}\right) \frac{d\lambda_r}{1+z} \frac{\delta t_e}{\delta t_r} \frac{1}{4\pi f^2(\chi)} \\
&= \frac{L}{(1+z)^2} \mathcal{I}\left(\frac{\lambda_r}{1+z}\right) d\lambda_r \frac{\delta t_e}{\delta t_r} \frac{1}{4\pi f^2(\chi)} \\
&= \frac{L}{(1+z)^3} \mathcal{I}\left(\frac{\lambda_r}{1+z}\right) d\lambda_r \frac{1}{4\pi f^2(\chi)}
\end{aligned}$$

The total bolometric flux is integrated over all wavelengths as

$$\begin{aligned}
\mathcal{F}^{\text{bol}} &= \int \mathcal{F}(\lambda_r)d\lambda_r \\
&= \int \frac{L}{(1+z)^3} \mathcal{I}\left(\frac{\lambda_r}{1+z}\right) \frac{1}{4\pi f^2(\chi)} d\lambda_r \\
&= \int \left(L \mathcal{I}\left(\frac{\lambda_r}{1+z}\right) d\frac{\lambda_r}{1+z} \right) \frac{1}{4\pi f^2(\chi)(1+z)^2} \\
&= \frac{L^{\text{bol}}}{4\pi \mathcal{D}_{\text{lum}}^2(\chi)}
\end{aligned}$$

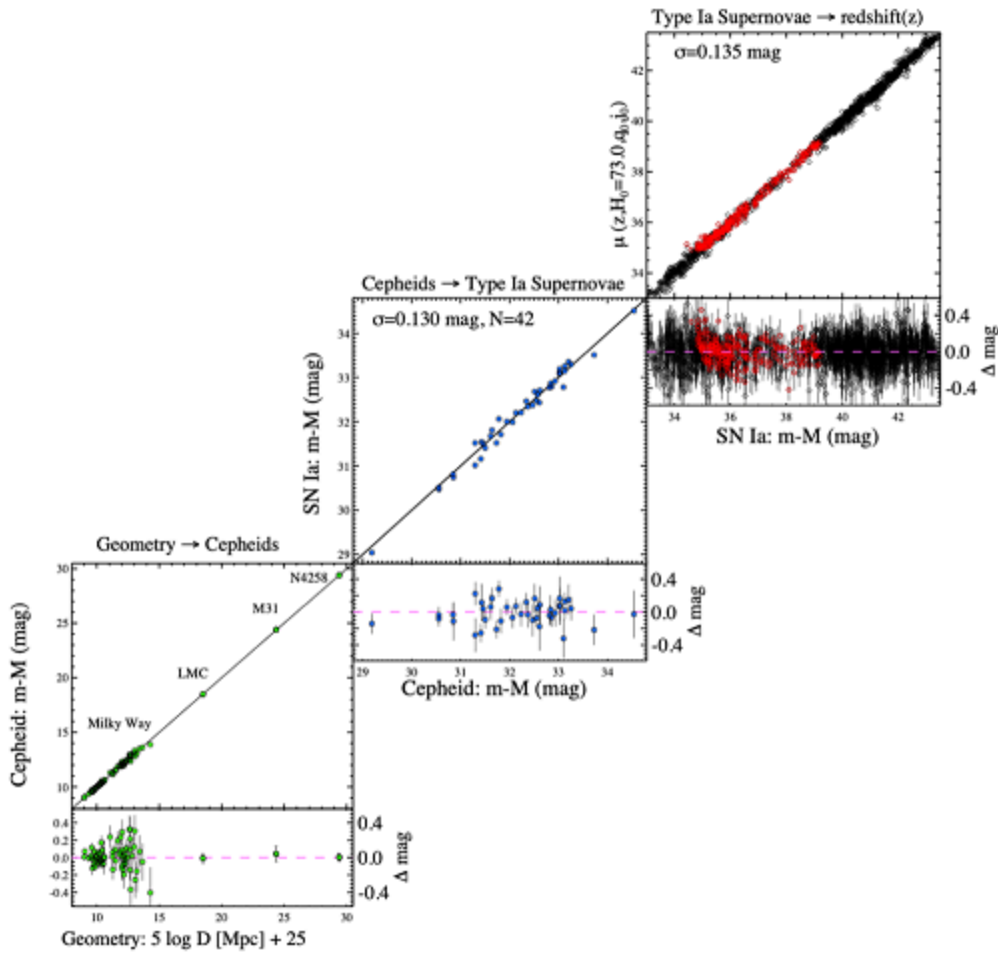
Hence the photons lose energy through the expansion of the universe, and we obtain the effective luminosity distance, as,

$$\mathcal{D}_{\text{lum}}(\chi) = \sqrt{\frac{L^{\text{bol}}}{4\pi \mathcal{F}^{\text{bol}}}} = f(\chi)(1+z)$$

Cosmic Distance Ladder, with objects of similar luminosity. *Radius of Earth* through historical shadow measurements at multiple places, *Distance of moon* based on lunar eclipse timings, *Distance to sun* through half phase angle between Moon and Sun or through Venus transit times, *Distance to nearby stars*, in terms of geometric parallax, all done sequentially, *Distance to galactic clusters*, using VLBI of active Masers or detached eclipsing binaries.

Cepheid Standard Candles, Period Luminosity relation for Cepheid variables. Estimating flux through oscillatory observations. Rapid oscillations. Type I and II Cepheids, error in Hubble observations.

Fixing measurements through *Type Ia Supernova Standard Candles*, recalibration with Cepheid variables. Cosmic Calibration.



Friedmann Equations

General Relativity

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

which can be simplified, as $G_{00} = \left(\frac{\dot{a}}{a}\right)^2 + 3\frac{\kappa c^2}{a^2}$, and $G_{ij} = -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa c^2}{a^2}\right]a^2\gamma_{ij}$ where γ_{ij} is the metric of the homogeneous and isotropic 3-space, i.e., $d\Sigma^2 = \gamma_{ij}dx^i dx^j$.

For an ideal perfect fluid $T_{\mu\nu} = (\rho c^2 + p)u_\mu u_\nu + pc^2 g_{\mu\nu}$, which has $T_{00} = \rho c^4$ and $T_{ij} = p(t)c^2 a^2 \gamma_{ij}$, and equating the relevant quantities, we have,

$$\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa c^2}{a^2}\right] = \frac{8\pi G \rho}{3}$$

$$-\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa c^2}{a^2}\right] = 8\pi G \frac{p}{c^2}$$

Combining the above two equations, we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\varrho + \frac{3p}{c^2} \right)$$

This shows that for energy density in the Universe with a non-negative pressure, the scale factor $a(t)$ cannot have positive acceleration.

Since the scale factor is a growth factor we are free to choose the normalization, such that $a(t_0) = 1$ at $t = t_0$ present. $H_0 = \frac{\dot{a}}{a}$ at the current time acts as the initial condition for the problem.

For spatially flat, we have

$$\varrho_{\text{critical}} = \frac{3H_0^2}{8\pi G}$$

and substituting in the general expression, we can rewrite as,

$$\left(\frac{\dot{a}}{a} \right)^2 = H_0^2 \left(\frac{\varrho}{\varrho_{\text{critical}}} - \frac{\kappa c^2}{a^2 H_0^2} \right)$$

and thereby,

$$-\frac{\kappa c^2}{H_0^2} = \left(1 - \frac{\varrho_0}{\varrho_{\text{critical}}} \right) = \Omega_\kappa$$

where the criticality is understood as the density between closed and open universe. Thus, the Friedmann equation is simplified, as

$$\left(\frac{\dot{a}}{a} \right)^2 = H_0^2 \left(\frac{\varrho}{\varrho_{\text{critical}}} + \frac{\Omega_\kappa}{a^2} \right)$$

For $p_{\text{matter}} = 0$, $p_{\text{rad}} = \frac{\varrho_{\text{rad}} c^2}{3}$, and relations $\varrho_{\text{matter}} \propto a^{-3}$ and $\varrho_{\text{rad}} \propto a^{-4}$, such that the matter density falls off, while the energy density of radiation changes due to the change in volume as well as the change in energy that a photon undergoes when the Universe expands or contracts.

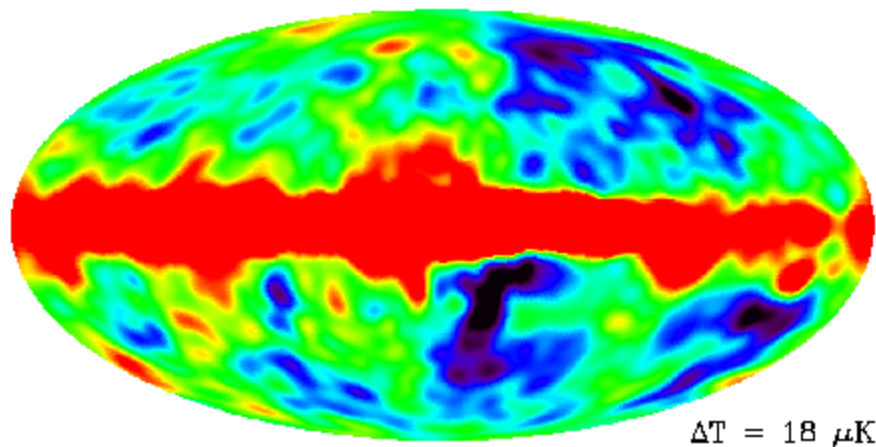
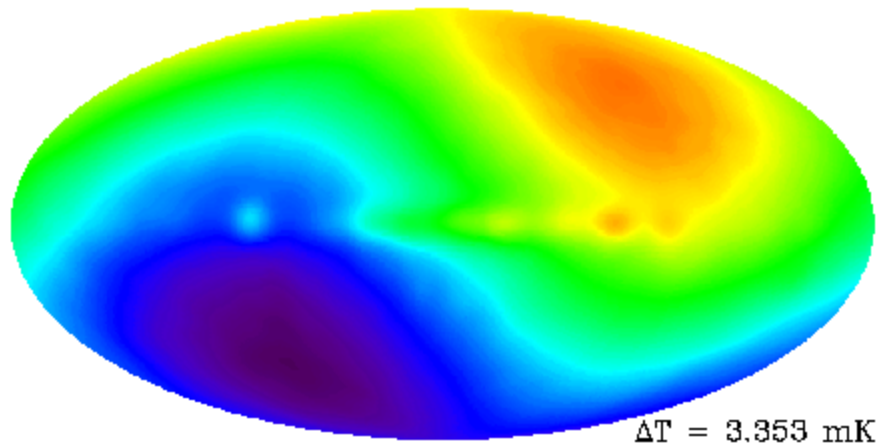
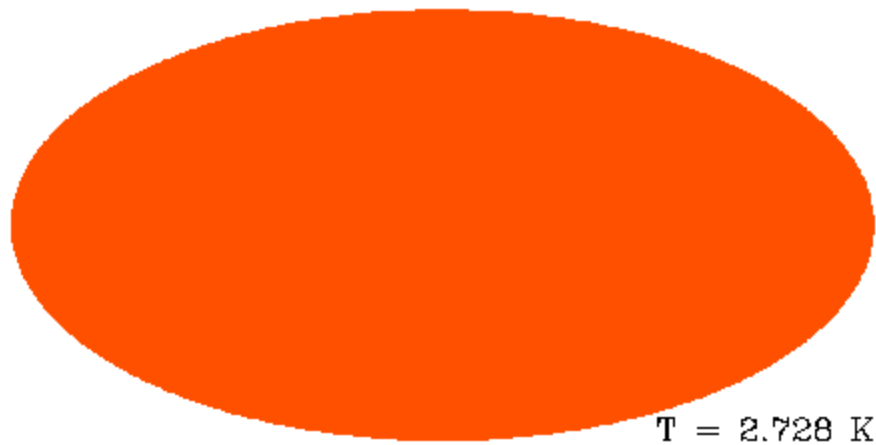
For the combined fluid densities with $\Omega_{\text{matter}} = \frac{\varrho_{\text{matter}}}{\varrho_{\text{critical}}}$, $\Omega_{\text{rad}} = \frac{\varrho_{\text{rad}}}{\varrho_{\text{critical}}}$, $\Omega_w = \frac{\varrho_w}{\varrho_{\text{critical}}}$ all defined at $t = t_0$, where Ω_w denotes the dark energy pressure with $p_w = w\varrho c^2$, such that, we have the combined equation as,

$$\left(\frac{\dot{a}}{a} \right)^2 = H_0^2 \left[\Omega_{\text{mat}} a^{-3} + \Omega_{\text{rad}} a^{-4} + \Omega_w a^{-3(1+w)} + \Omega_\kappa a^{-2} \right] = H_0^2 E^2(a)$$

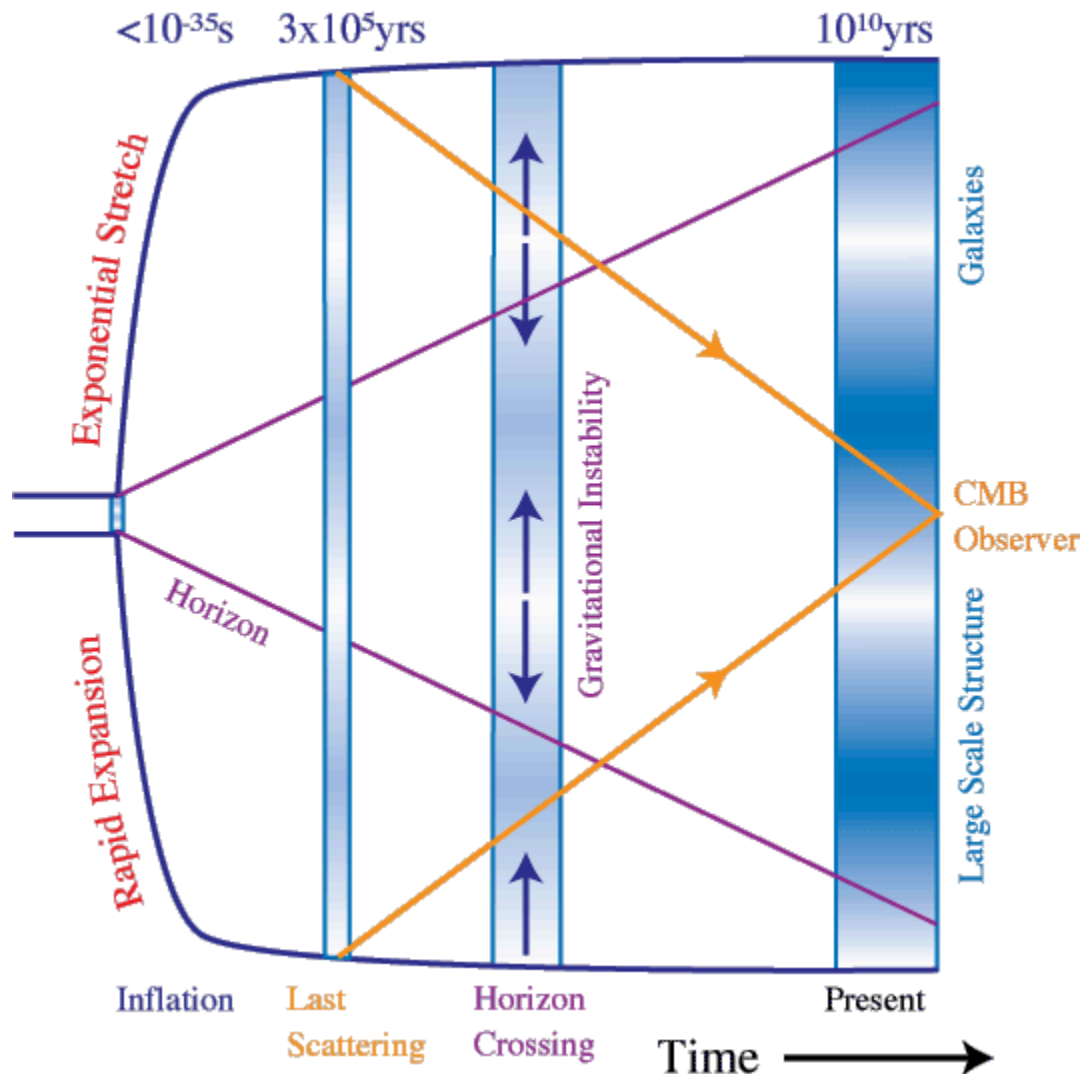
where $\Omega_\kappa = 1 - \Omega_{\text{matter}} - \Omega_{\text{rad}} - \Omega_w$. Computing $a(t)$, thereby $\chi = d\chi = \frac{dt}{a(t)}$, thereby, computing the age of the universe.

L02.01 Cosmography using GWs

Cosmic Microwave Background

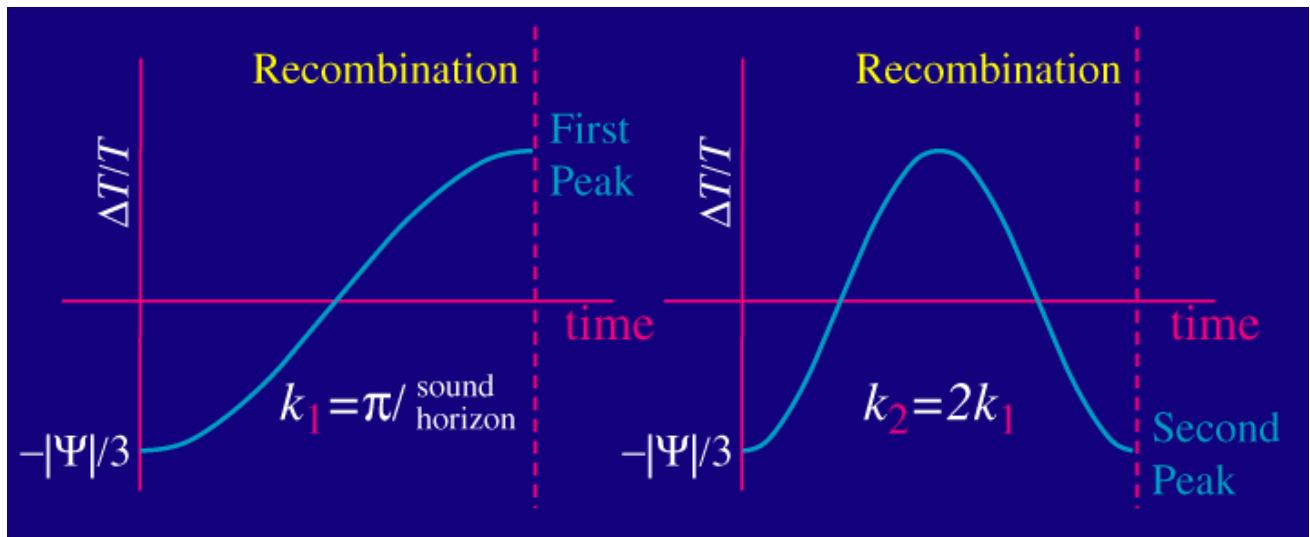


- Extremely isotropic emission, evidence of anisotropies at $O(10^{-3})$ Kelvins.
- Dipole corresponds to peculiar velocity with respect to CMB, due to Doppler shifts.
- Fluctuations observed only at the level of 1 in 10^5 , seed fluctuations in structure formation.
- Smaller scale, sound propagation due to pressure waves. Photon fluctuation in baryon potential wells. Density functional decomposition into a Fourier spectrum.

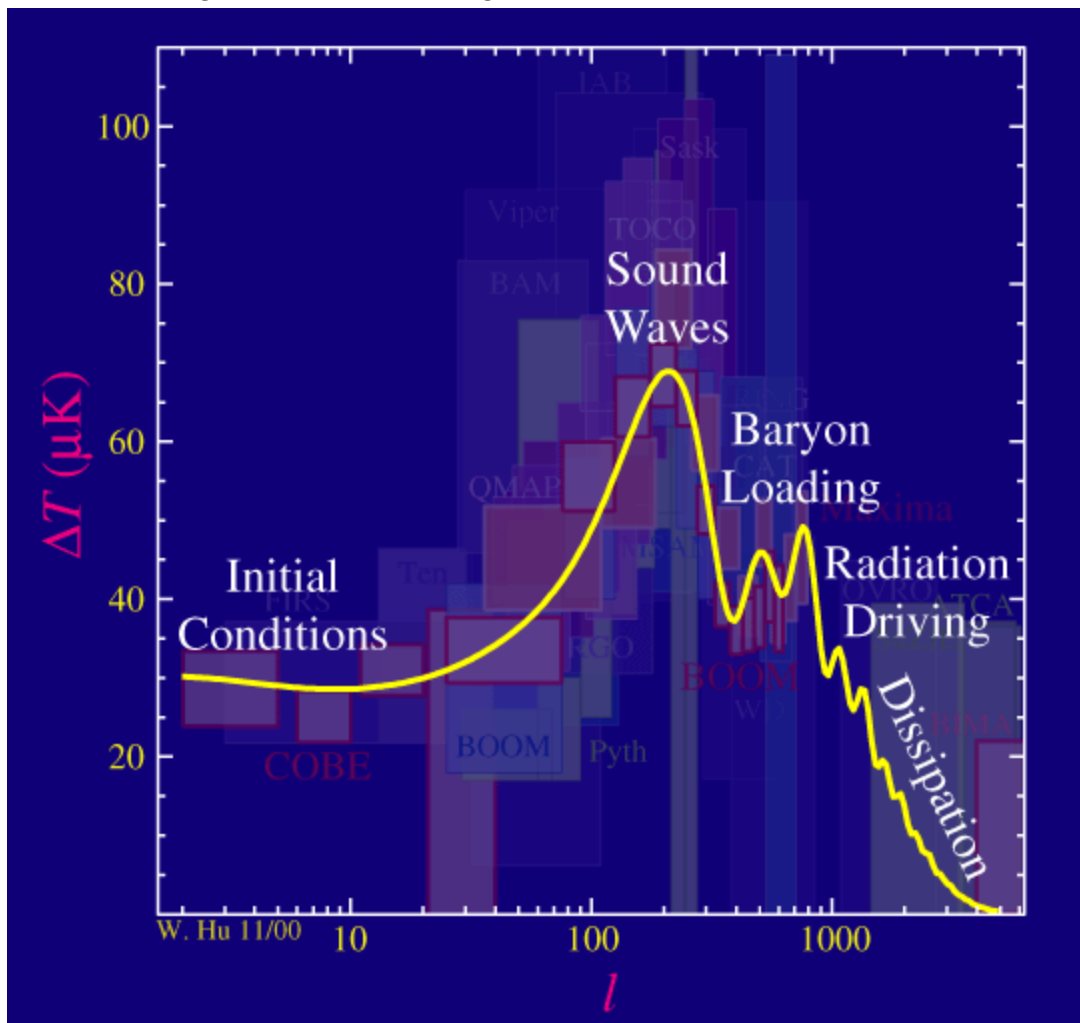


- In the very early universe, we think a period of rapid expansion, called *inflation*, caused these quantum fluctuations to be stretched into cosmic scales. These fluctuations in the energy density imply fluctuations in the local gravitational potential. Regions of high density generate potential wells. Regions of low density generate potential hills.
- Sound waves stop oscillating at recombination when the baryons release the photons. Modes that reach extrema of their oscillation (maximal compression or rarefaction in

potential wells) by recombination will carry enhanced temperature fluctuations.



- Power spectral density can help constrain parameters in the Hubble dynamics, including Ω_κ , since the angular diameter changes.

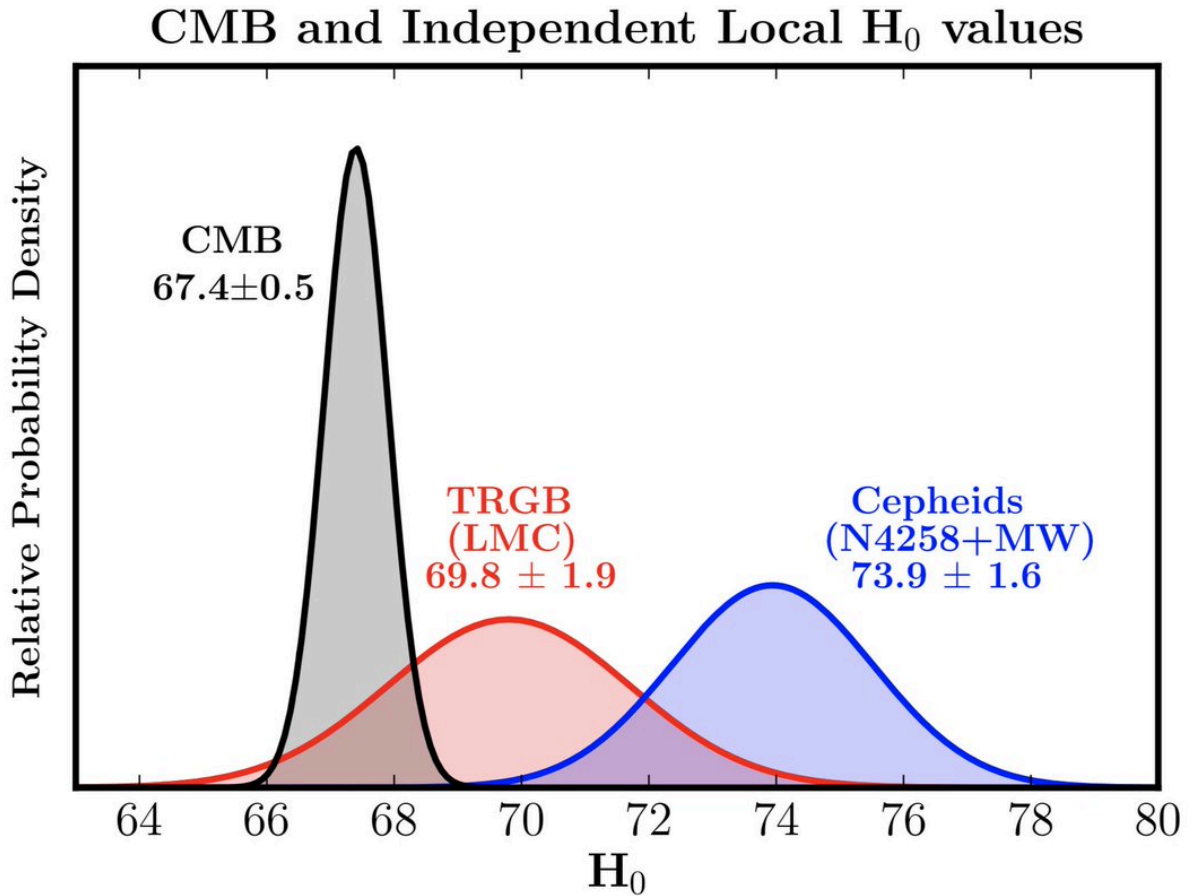


Angular wavenumber, called a multipole ℓ , of the power spectrum is related to the inverse of the angular scale.

- Ratios of alternate peaks, comprising maximal rarefaction and maximal compression, helping constrain cosmological parameters.

Hubble Tension

For Λ CDM universe, there is a discrepancy between Baseline standard candle results and CMB inferences.



Expectation to resolve using gravitational waves.

Primer (*Semi-Classical*) on GW

Frequency Evolution

For two bodies of mass M and m , separated from centre of mass by R and r , we have $M\omega^2 R = \frac{GMm}{(R+r)^2}$ and $m\omega^2 r = \frac{GMm}{(R+r)^2}$, and we have the relation $MR = mr$, thereby, we arrive at

$$\omega^2 = \frac{G(M+m)}{(R+r)^3}$$

and the total energy E_{tot} is,

$$E_{\text{tot}} = -\frac{1}{2} \frac{GMm}{R+r} = -\frac{1}{2} \frac{G^{\frac{2}{3}} Mm}{(M+m)^{\frac{1}{3}}} \omega^{\frac{2}{3}}$$

The power $\mathcal{P}_{\text{rad}}$ radiated is proportional to the quadropole moment $\mathcal{Q}$, related to the square of

the moment of inertia $\mathcal{I} = mr^2 + MR^2$, thereby, simplifying, we have,

$$\mathcal{I} = \frac{mM}{m+M} (R+r)^2$$

related to the reduced mass ratio $\mu = \frac{mM}{m+M}$, and thereby,

$$\mathcal{P}_{\text{rad}} \propto \frac{G\mathcal{I}^2\omega^6}{c^3}$$

and now noting the rate of change of total energy can be related as,

$$-\frac{d}{dt}E_{\text{tot}} = \frac{1}{3}G^{\frac{2}{3}} \frac{mM}{(M+m)^{\frac{1}{3}}} \omega^{-\frac{1}{3}} \frac{d\omega}{dt}$$

and by the effective loss of energy, we have, $\mathcal{P}_{\text{rad}} = -\frac{d}{dt}E_{\text{tot}}$, such that,

$$\alpha G \frac{m^2 M^2}{(M+m)^2} \left(\frac{G(M+m)}{\omega^2} \right)^{\frac{4}{3}} \frac{\omega^6}{c^3} = \frac{1}{3} G^{\frac{2}{3}} \frac{mM}{(M+m)^{\frac{1}{3}}} \omega^{-\frac{1}{3}} \frac{d\omega}{dt}$$

Hence, we have,

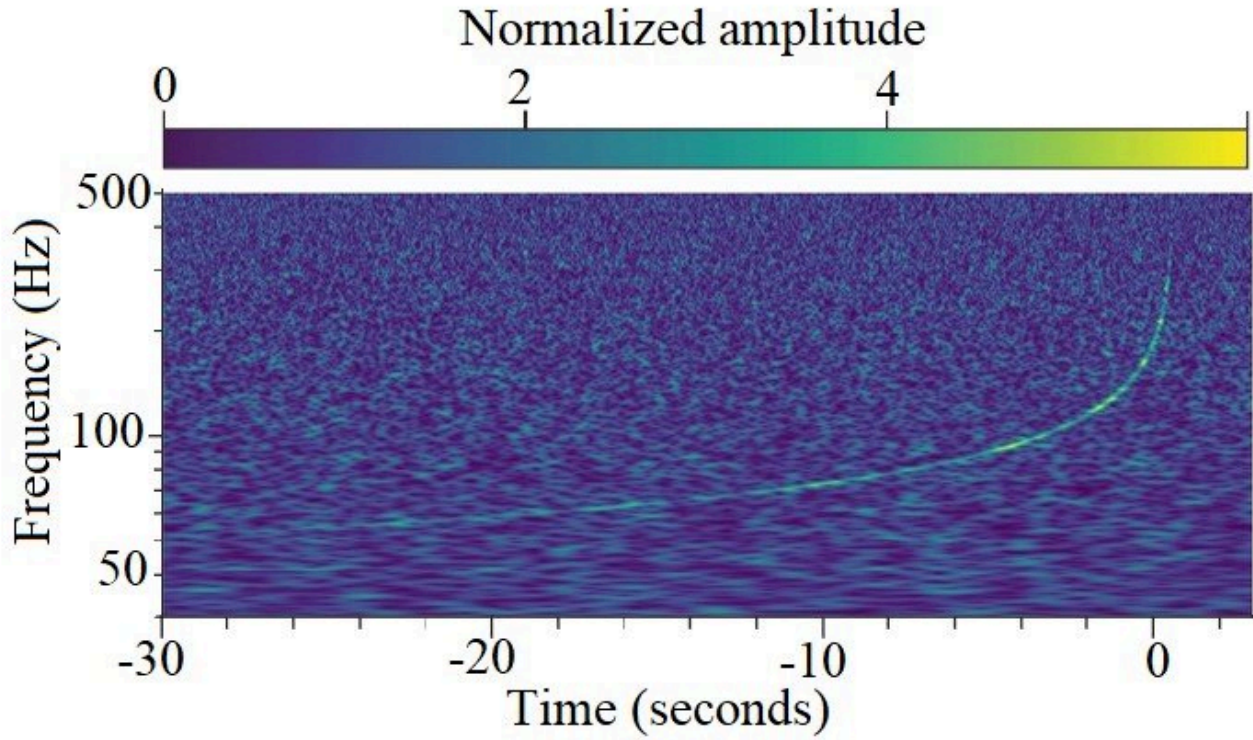
$$\frac{d\omega}{dt} = \alpha \frac{G^{\frac{5}{3}}}{c^3} \frac{mM}{(m+M)^{\frac{1}{3}}} \omega^{\frac{11}{3}}$$

Thereby

$$\mathcal{M} = \frac{(mM)^{\frac{3}{5}}}{(m+M)^{\frac{1}{5}}} = \frac{c^3}{G} \left(\frac{1}{3\alpha} \omega^{-\frac{11}{3}} \frac{d\omega}{dt} \right)^{\frac{3}{5}}$$

with the effective mass combination as the chirp mass $\mathcal{M}$. Note that the gravitational wave frequency f_{GW} is related by,

$$2\pi f_{\text{GW}} = \omega_{\text{GW}} = 2\omega$$



For cosmological expansion of the universe, we have to account for the redshifted frequencies, thereby, we have the relations

$$\omega_{\text{det}} = \frac{\omega}{1+z} \quad t_{\text{det}} = \frac{t}{1+z} \quad \mathcal{M}_{\text{det}} = \mathcal{M}(1+z)$$

Degeneracy in the frequency evolution due to the detector frame masses.

Signal Amplitudes

Gravitational wave polarisations in frequency domain,

$$\begin{aligned} \tilde{h}_+(f) &= \mathcal{A}(f, \mathcal{M}) \left(\frac{1 + \cos^2 \iota}{2} \right) \exp(i\Psi(f, \mathcal{M})) \\ \tilde{h}_\times(f) &= \mathcal{A}(f, \mathcal{M}) \cos \iota \exp\left(\frac{\pi}{2} + i\Psi(f, \mathcal{M})\right) \end{aligned}$$

where the amplitude and phases are given by

$$\begin{aligned} \mathcal{A}(f, \mathcal{M}) &= \frac{1}{\mathcal{D}_L} \frac{5}{24\pi^{\frac{4}{3}}} \frac{(GM)^{\frac{5}{6}}}{c^{\frac{3}{2}}} \frac{1}{f^{\frac{7}{6}}} \\ \Psi(f, \mathcal{M}) &= 2\pi f t_m - \frac{\pi}{4} - \phi_c + \frac{3}{128} \left(\frac{\pi G \mathcal{M}}{c^3} \right)^{-\frac{5}{3}} \frac{1}{f^{\frac{5}{3}}} \end{aligned}$$

The amplitude can give the luminosity distance $\mathcal{D}_L$, which can be corroborated for the redshift, and give cosmological parameters.

Sky localisation with multiple interferometers, by measuring the delay in respective time of coalescences. Response depends on detectors with the antenna pattern function, $F_{+, \times}(\alpha, \theta, \psi, \iota)$.

Binary Evolution (Newtonian Analysis)

Given at leading order, gravitational waves are generated by a time-varying mass quadrupole moment,

$$h_{ij}^{\text{TT}}(\mathbf{x}) = \frac{2G}{c^4 r} \ddot{Q}_{ij}^{\text{TT}}(t - t_0)$$

where r is the distance from source to observer, $t_{\text{ret}} = t - \frac{r}{c}$ is the retarded time, and Q_{ij} is the *mass quadrupole moment*, given by

$$Q^{ij} = M^{ij} - \frac{1}{3} \delta^{ij} M^k_k$$

where M^{ij} is the *second mass moment* defined as

$$M^{ij}(t) = \int d^3x \varrho(\mathbf{x}, t) x^i x^j$$

Now, we consider a binary system with m_1 and m_2 in a circular orbit with velocity ω and relative separation R .

For the origin at the centre of mass, we compute the position vectors,

$x_1^i = (\frac{m_2}{M} R \cos \omega t_{\text{ref}}, \frac{m_2}{M} R \sin \omega t_{\text{ref}}, 0)$, $x_2^i = (-\frac{m_1}{M} R \cos \omega t_{\text{ref}}, -\frac{m_1}{M} R \sin \omega t_{\text{ref}}, 0)$, we have the second moment tensor $M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j$ as

$$M^{ij} = \frac{1}{2} \mu R^2 \begin{pmatrix} \cos 2\omega t_{\text{ref}} + 1 & \sin 2\omega t_{\text{ref}} & 0 \\ \sin 2\omega t_{\text{ref}} & -(\cos 2\omega t_{\text{ref}} - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thereby, we have

$$\ddot{M}^{ij} = 2\mu R^2 \omega^2 \begin{pmatrix} -\cos 2\omega t_{\text{ref}} & -\sin 2\omega t_{\text{ref}} & 0 \\ -\sin 2\omega t_{\text{ref}} & \cos 2\omega t_{\text{ref}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We now use

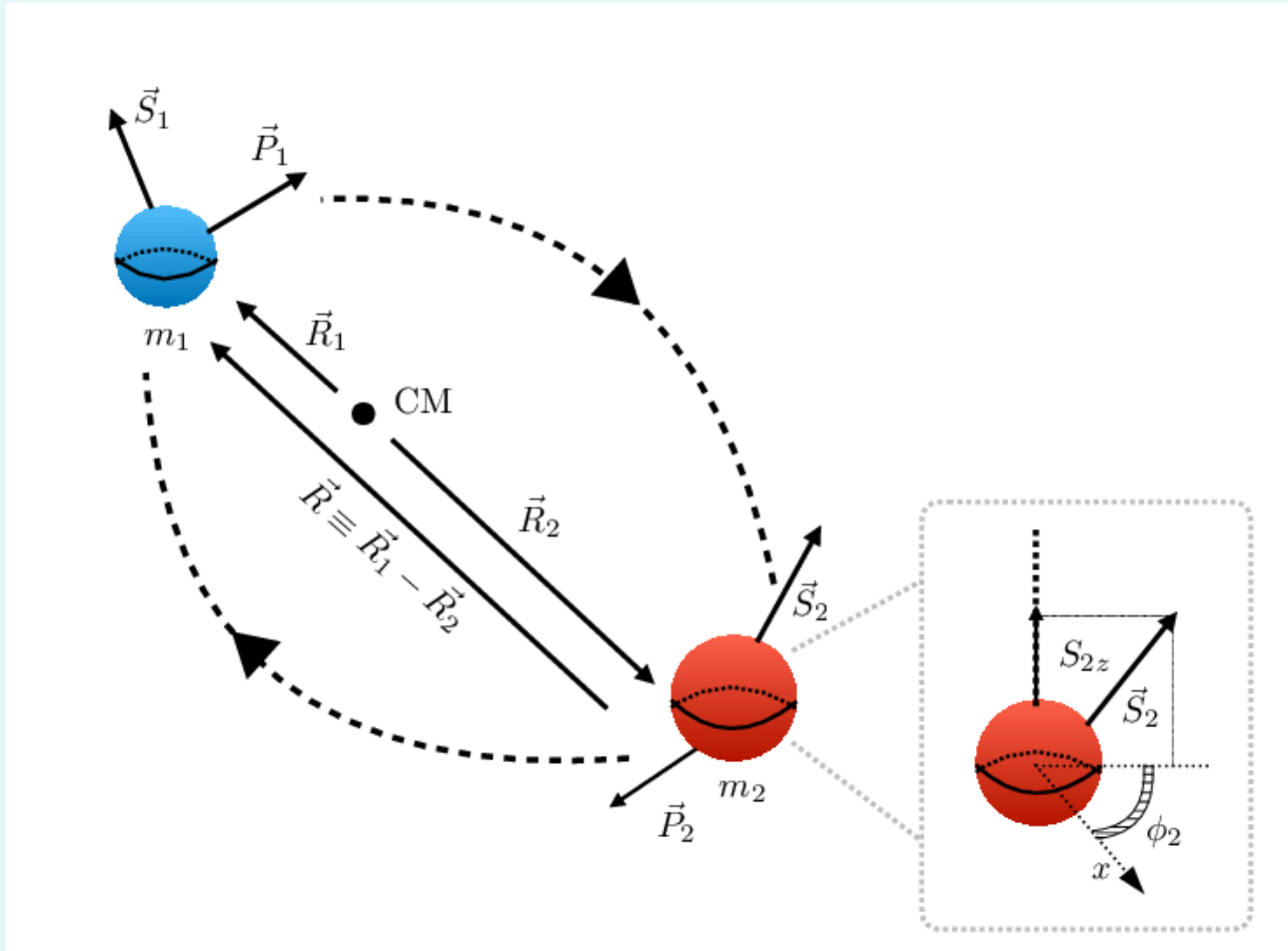
$$h_+ = \frac{G}{c^4 r} (\ddot{M}^{11} - \ddot{M}^{22})$$

$$h_{\times} = \frac{2G}{c^4 r} \ddot{M}^{12}$$

to arrive at

$$h_+ = -\frac{4G}{c^4 r} \mu R^2 \omega^2 \cos 2\omega t_{\text{ref}}$$

$$h_\times = -\frac{2G}{c^4 r} \mu R^2 \omega^2 \sin 2\omega t_{\text{ref}}$$



Now we use the Einstein quadrupole formula, relating the total power $\mathcal{P}$ radiated away,

$$\mathcal{P} = \frac{G}{5c^5} \langle \ddot{Q}^{ij} \ddot{Q}_{ij} \rangle$$

Since, for our example, we have $\ddot{Q}^{ij} = \ddot{M}^{ij}$. thereby,

$$\mathcal{P} = \frac{G}{5c^5} \langle \ddot{M}^{ij} \ddot{M}_{ij} \rangle$$

where we have,

$$\ddot{M}^{ij} = 4\mu R^2 \omega^3 \begin{pmatrix} \sin 2\omega t_{\text{ref}} & -\cos \omega t_{\text{ref}} & 0 \\ -\cos 2\omega t_{\text{ref}} & -\sin 2\omega t_{\text{ref}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence, contracting,

$$\ddot{M}^{ij}\ddot{M}_{ij} = 32\mu^2 R^2 \omega^2$$

hence the power radiated simplifies as

$$\mathcal{P} = \frac{16G\mu^2 R^2 \omega^6}{5c^5}$$

We can use Kepler's third law $\omega^2 = \frac{GM}{R^3}$, and set

$$\mathcal{A} = \frac{4G}{c^4 r} \mu R^2 \omega^2$$

Further, setting $2\pi f_{\text{GW}} = 2\omega$, we have the expressions,

$$h_+ = -\frac{4}{r} \left(\frac{GM}{c^2} \right)^{\frac{5}{3}} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{\frac{2}{3}} \cos 2\pi f_{\text{GW}} t_{\text{ref}}$$

$$h_\times = -\frac{4}{r} \left(\frac{GM}{c^2} \right)^{\frac{5}{3}} \left(\frac{\pi f_{\text{gw}}}{c} \right)^{\frac{2}{3}} \sin 2\pi f_{\text{GW}} t_{\text{ref}}$$

where we have the chirp mass $\mathcal{M} = \frac{(m_1 m_2)^{\frac{3}{5}}}{(m_1 + m_2)^{\frac{1}{5}}}$. We express the power radiated as

$$\mathcal{P} = \frac{32c^5}{5G} \left(\frac{\pi G \mathcal{M} f_{\text{GW}}}{c^3} \right)^{\frac{10}{3}}$$

and the total energy $E = -\frac{Gm_1 m_2}{2R}$ can be derived as

$$E = -\left(\frac{\pi^2 G^2 \mathcal{M}^2 f_{\text{GW}}^2}{8} \right)^{\frac{1}{3}}$$

Now we use the energy balance $\mathcal{P} = -\dot{E}$, the rate of loss of energy from the system is equal to the power radiated in GWs, then the rate of increase of the GW frequency, can be seen as

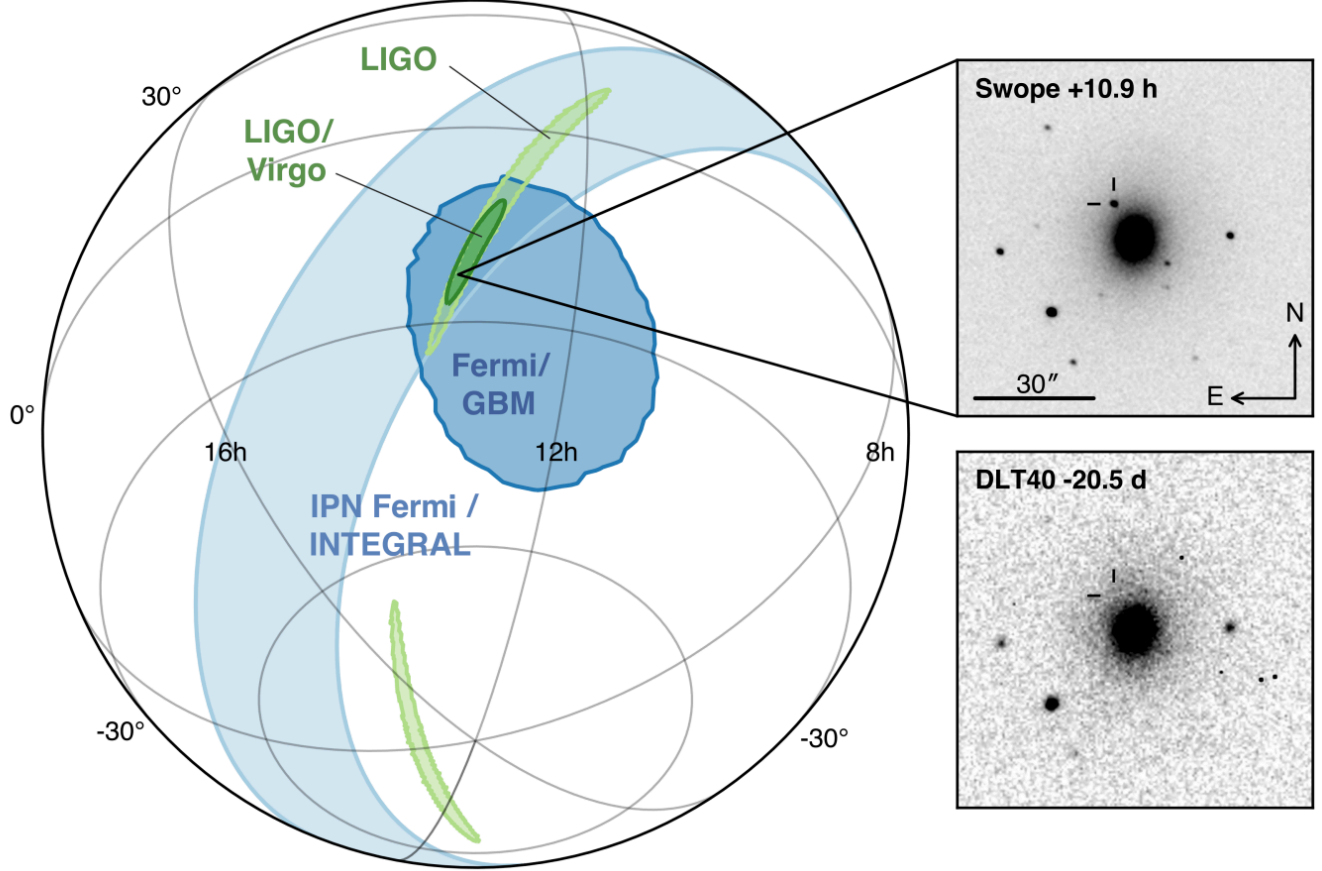
$$\frac{32c^5}{5G} \left(\frac{\pi G \mathcal{M} f_{\text{GW}}}{c^3} \right)^{\frac{10}{3}} = \frac{1}{3} \left(\frac{\pi^2 G^2 \mathcal{M}^2}{f_{\text{GW}}} \right)^{\frac{1}{3}} \dot{f}_{\text{GW}}$$

to arrive at

$$\dot{f}_{\text{GW}} = \frac{96c^5 \pi^{\frac{8}{3}}}{5G} \left(\frac{G \mathcal{M}}{c^3} \right)^{\frac{5}{4}} f_{\text{GW}}^{\frac{11}{3}}$$

L03.01 Gravitational Wave Cosmology

Standard Siren: GW170814



Follow up analysis with electromagnetic counterpart. Constrained redshift from spectra observations, of **NGC 4993** and kilonova emissions. This further constrains the Hubble constant, $v = H_0 d$, leading to posterior for the Hubble constant. Account must be taken to separate out the recession and peculiar velocities of the source galaxy, and considering the GW wave velocity.

Bayesian Approach

Likelihood, given our dataset of GW observation, recession velocity v_r , *peculiar velocity* averaged v_p given the inference parameters,

$$\mathcal{L}(x_{\text{GW}}, v_r, \langle v_p \rangle | d, \cos \iota, v_p, H_0) = p(x_{\text{GW}} | d, \cos \iota) p(v_r | d, v_p, H_0) p(\langle v_p \rangle | v_p)$$

where we have the cosmological v_r can be assumed normal distribution, with

$$p(v_r | d, v_p, H_0) \sim \mathcal{N}(v_p + H_0 d, \sigma_{v_r}^2)(v_r)$$

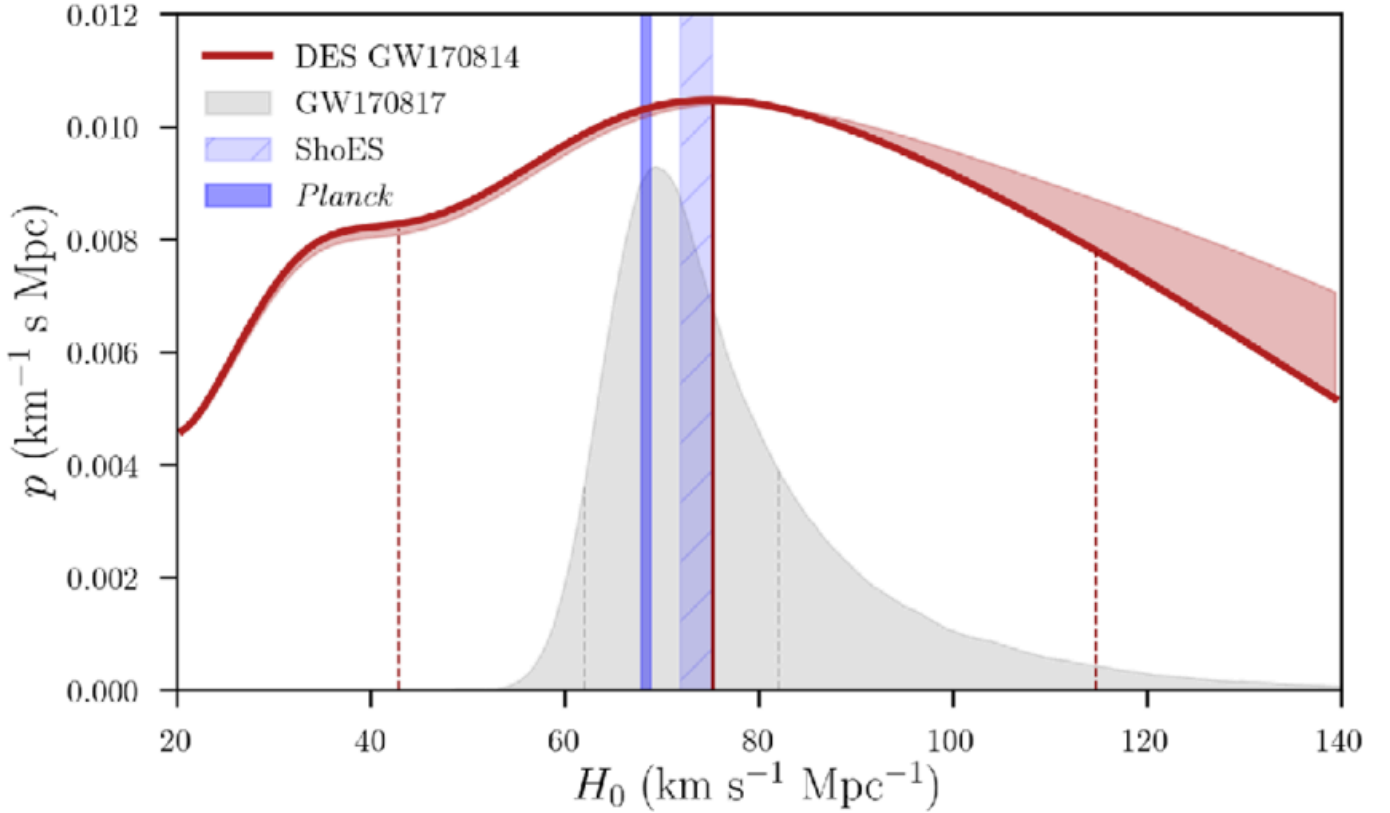
and similarly,

$$p(\langle v_p \rangle | v_p) \sim \mathcal{N}(v_p, \sigma_{v_p}^2)(v_p)$$

such that, we have the posterior,

$$p(H_0, \{d, \cos \iota, v_p\} | x_{\text{GW}}, v_r, \langle v_p \rangle) \propto p(x_{\text{GW}} | d, \cos \iota) p(v_r | d, v_p, H_0) p(\langle v_p \rangle | v_p) \frac{\pi(H_0)}{\mathcal{N}_s(H_0)} \pi(d) \pi(v_p) \pi(\cos \iota)$$

where the $\mathcal{N}_s(H_0)$ accounts for the selection criteria. Marginalizing over the other parameters, we obtain the posterior $p(H_0)$.



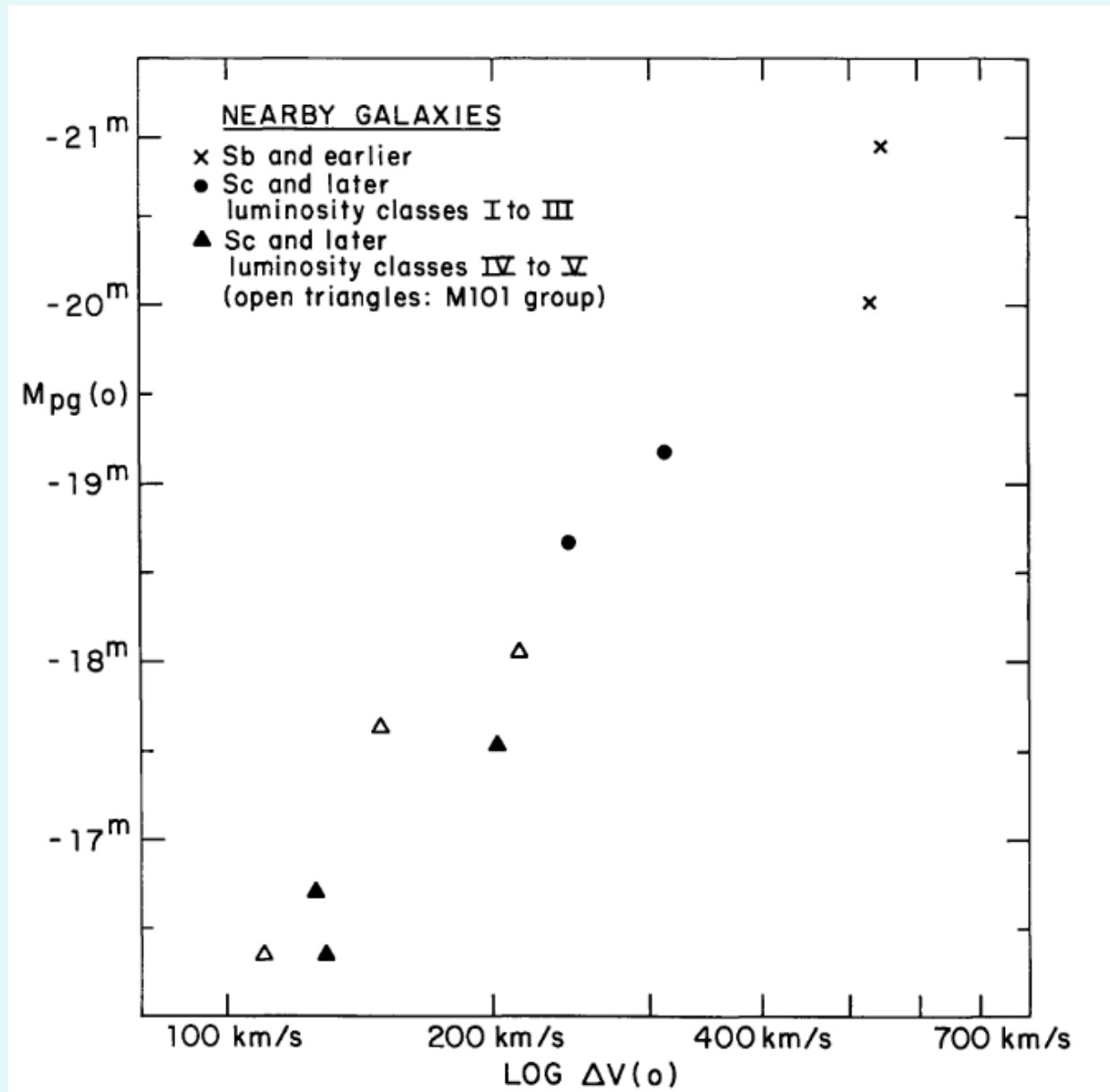
Follow up by observations of *apparent* superluminal jets perpendicular to inclination, by VLBI (Very Long Baseline Interferometry). Measurement breaks degeneracy between distance $\mathcal{D}_L$ and inclination ι .

Peculiar Velocities Measurements

Galaxy Scaling Relations and their Residuals:

- Tully Fisher Relation: Strongly constrains the luminosity and *rotational* velocity of galaxies, residuals in the fundamental plane are correlated with peculiar velocities.

Highly constrained relation, with minimal scatter.



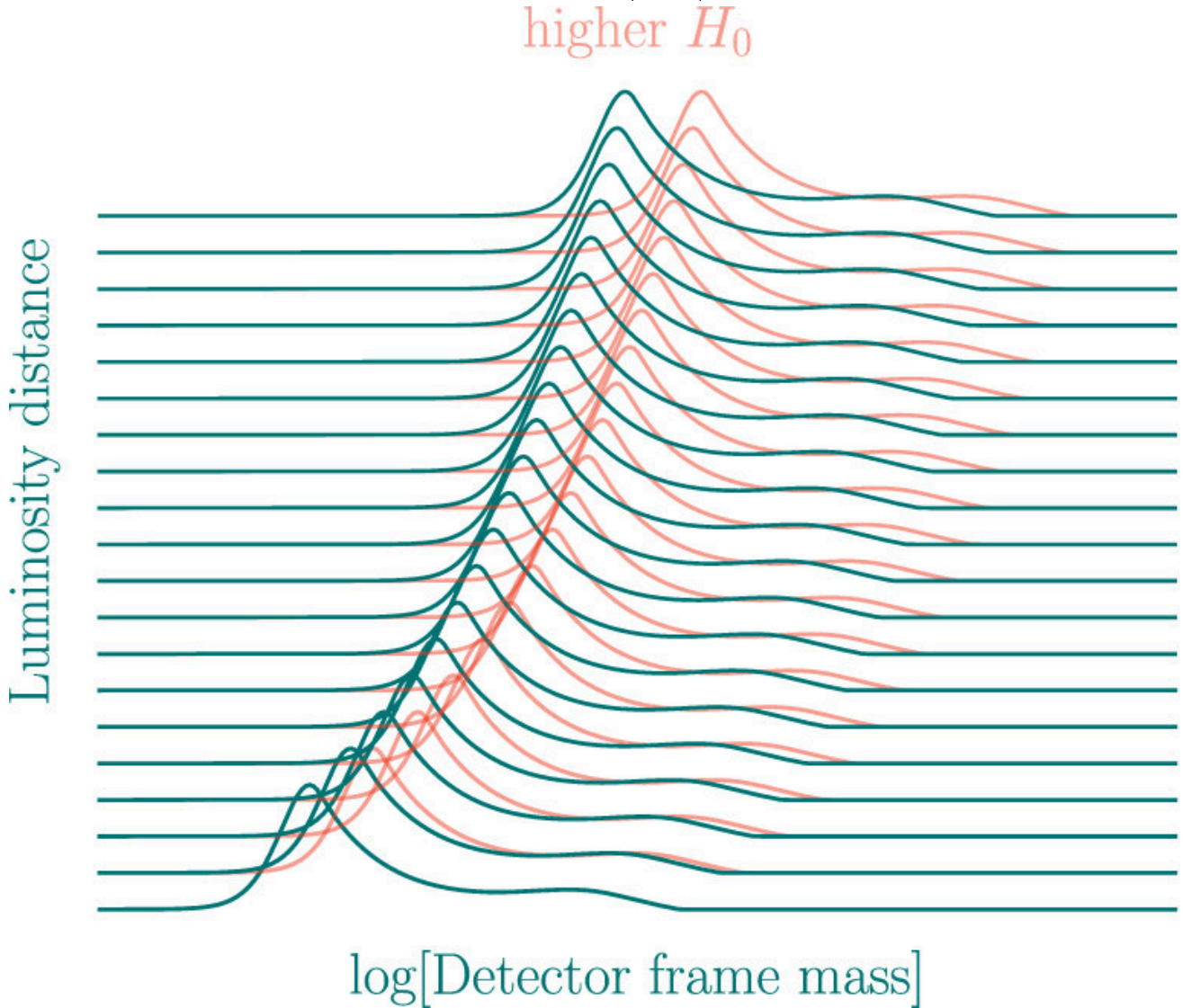
- The fundamental plane, is a relation between size, surface brightness and velocity dispersion. Correlated with the *Virial Theorem*.

Hubble constant from BBHs

Determining Hubble constant from GW observations with no electromagnetic counterpart. We can estimate $H_0 < H_{\max}$, then the error box can be surveyed for bright galaxies, with velocities below $v < H_{\max} r$. Statistical analysis, through analyzing clusters in the localisation determined earlier, by understanding their redshifts.

Statistical Host Identification, ignoring the clustering of galaxies, potential host is in the galaxy catalog, with the localization, using redshift measurements. Single galaxy with the *correct* Hubble constant, rest are randomly distributed. Corroborated with multiple events.

Spectral Sirens, Invariance of mass distribution of BBH merger events, while GW events measure the detector frame quantities, $\mathcal{M} \rightarrow \mathcal{M}(1+z)$.



Shifts in the mass distribution with redshifts, constrains the Hubble constant. For increasing luminosity distance, the whole distribution of detector frame masses shifts to higher values. The amount shifted corresponds to the redshift at a given luminosity distance and it is therefore sensitive to the expansion rate. For example, a higher value of H_0 associates higher detected masses to the same distance.

Bayesian approach must be implemented, necessary since there are huge degeneracies in the intrinsic mass measurements. For cosmological parameters, rate and shape parameters, we have $\Lambda = \{\Omega, \lambda', \mathcal{N}\}$, such that,

$$p(\Lambda|\{d\}) = \frac{p(\{d\}|\Lambda)p(\Lambda)}{p(\{d\})}$$

where we have $\frac{d\mathcal{N}}{d\theta}(\Lambda) = \mathcal{N}p_{\text{pop}}(\theta, \lambda')$ as the rate and shape parameters. Parameters of

individual events drawn from sample properties,

$$p(\{\theta\}|\lambda') = \prod_{i=1}^{N_{\text{obs}}} \frac{p_{\text{pop}}(\theta_i|\lambda')}{\int d\theta p_{\text{pop}}(\theta_i|\lambda')}$$

Even in the absence of error bars, we have to be careful about selection effects, where we weight by the detection probability,

$$p(\{\theta\}|\lambda') = \prod_{i=1}^{N_{\text{obs}}} \frac{p_{\text{pop}}(\theta_i|\lambda')p_{\text{det}}(\theta_i)}{\int d\theta p_{\text{pop}}(\theta_i|\lambda')p_{\text{det}}(\theta)} = \prod_{i=1}^{N_{\text{obs}}} \frac{p_{\text{pop}}(\theta_i|\lambda')}{\int d\theta p_{\text{pop}}(\theta_i|\lambda')p_{\text{det}}(\theta)}$$

Further, incorporating noise as series of realizations and an overall threshold,

$$p_{\text{det}}(\theta) = \int_{\text{Threshold}} p(\{d\}|\theta)d\theta = \int \mathbf{I}_{\{d\}} p(\{d\}|\theta)d\theta$$

where the indicator function selects the detectable events.

We have the probability of the data given the parameters as

$$p(\{d\}|\lambda') = \frac{\int d\theta p(\{d\}|\{\theta\})p_{\text{pop}}(\theta|\lambda')}{\mathcal{Z}(\lambda')}$$

The evidence simplifies,

$$\begin{aligned} \mathcal{Z}(\lambda') &= \int d\{d\} \int d\theta p(\{d\}|\{\theta\})p_{\text{pop}}(\{\theta\}|\lambda') \\ &= \int d\theta \left(\int d\{d\} p(\{d\}|\{\theta\}) \right) p_{\text{pop}}(\{\theta\}|\lambda') \\ &= \int d\theta p_{\text{det}}(\{\theta\})p_{\text{pop}}(\{\theta\}|\lambda') \end{aligned}$$

L04.01 Gravitational Wave Cosmology

Bayesian Analysis

Shape Parameter

Recapping,

$$p(\{\theta\}|\lambda') = \prod_{i=1}^{N_{\text{obs}}} \frac{p_{\text{pop}}(\theta_i|\lambda') p_{\text{det}}(\theta_i)}{\int d\theta p_{\text{pop}}(\theta_i|\lambda') p_{\text{det}}(\theta)} = \prod_{i=1}^{N_{\text{obs}}} \frac{p_{\text{pop}}(\theta_i|\lambda')}{\int d\theta p_{\text{pop}}(\theta_i|\lambda') p_{\text{det}}(\theta)}$$

We have the probability of the data given the parameters as

$$p(\{d\}|\lambda') = \frac{\int d\theta p(\{d\}|\{\theta\}) p_{\text{pop}}(\theta|\lambda')}{\int d\theta p_{\text{det}}(\{\theta\}) p_{\text{pop}}(\{\theta\}|\lambda')}$$

Prior to the meta analysis, parameter estimation runs provide posteriors from the GW analysis. We have

$$p(d_i|\theta_i) = \frac{p(\theta_i|d_i) p(d_i)}{\pi(\{\theta\})}$$

where the priors are reweighted to those included in the likelihood. We thereby, have,

$$p(\{d\}|\lambda') = \prod_{i=1}^{N_{\text{obs}}} \frac{\frac{1}{s_i} \sum_{j=1}^{s_i} p_{\text{pop}}(\theta_i^j|\lambda') \frac{p(d_i)}{\pi(\theta)}}{\int d\theta p_{\text{det}}(\{\theta\}) p_{\text{pop}}(\{\theta\}|\lambda')}$$

such that we can use Bayesian theorem to arrive at,

$$p(\lambda'|\{d\}) = \frac{\pi(\lambda')}{p(\{d\})} \prod_{i=1}^{N_{\text{obs}}} \frac{\frac{1}{s_i} \sum_{j=1}^{s_i} p_{\text{pop}}(\theta_i^j|\lambda') \frac{p(d_i)}{\pi(\theta)}}{\int d\theta p_{\text{det}}(\{\theta\}) p_{\text{pop}}(\{\theta\}|\lambda')}$$

Rate Parameter

We assume a Poisson likelihood,

$$p(N_{\text{obs}}) \sim \exp(-N_{\text{det}}) N_{\text{det}}^{N_{\text{obs}}}$$

such that, we have the average number of observations

$$N_{\text{obs}} = \int \alpha p(N_{\text{obs}}) dN_{\text{obs}} = \int \alpha \exp(-N_{\text{det}}) N_{\text{det}}^{N_{\text{obs}}} = \alpha N_{\text{det}}$$

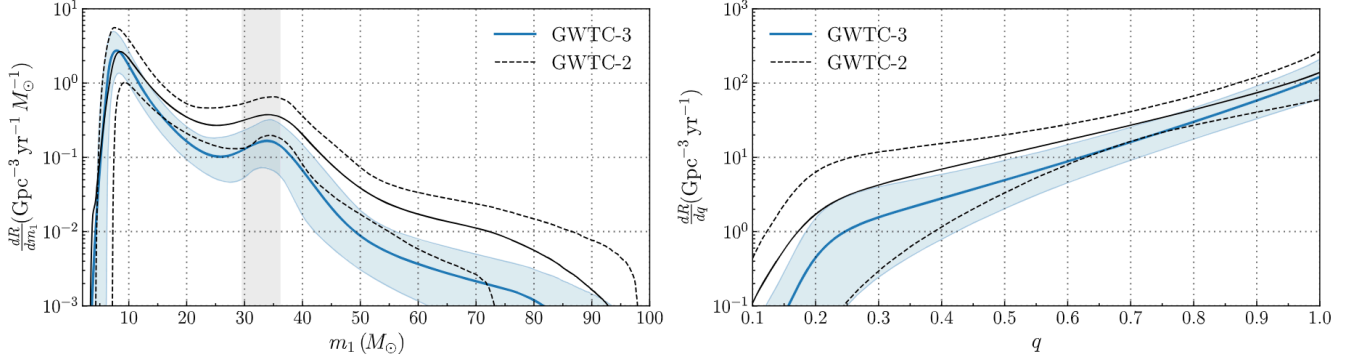
as the effective detection distribution.

Combined Likelihood

We have the overall likelihood

$$\mathcal{L}(\lambda', N|\{d\}) = \underbrace{\pi(\lambda')\pi(N)}_{\text{Prior}} \prod_{i=1}^{N_{\text{obs}}} \underbrace{\frac{\frac{1}{s_i} \sum_{j=1}^{s_i} p_{\text{pop}}(\theta_i^j|\lambda') \frac{p(d_i)}{\pi(\theta)} }{\int d\theta p_{\text{det}}(\{\theta\}) p_{\text{pop}}(\{\theta\}|\lambda')}}_{\text{Likelihood Samples}} \underbrace{\exp(-N_{\text{det}}) N_{\text{det}}^{N_{\text{obs}}}}_{\text{Number Parameters}}$$

For a fixed cosmology, we have the distributions of the posteriors.



Incorporating Redshift

We can further separate the cosmological parameters $\{\theta\} \equiv \{\theta', z\}$ by using independent priors on θ' and the line of sight redshift z prior. The redshift of the GW event could be related to galaxies with apparent magnitude m , absolute magnitude M , and that are either in the catalog (G), or not ($\tilde{G}$). We can marginalize,

$$\begin{aligned} p(z|\theta', \Lambda) &= \int \int \sum_{g \in G, \tilde{G}} dm dM p(z, m, M, g|\theta', \Lambda) \\ &= p(G|\theta', \Lambda) \underbrace{\int \int dm dM p(z, M, m|G, \Lambda)}_{\text{In Catalog}} + p(\tilde{G}|\theta', \Lambda) \underbrace{\int \int dm dM p(z, M, m|\tilde{G}, \Lambda)}_{\text{Out Catalog}} \end{aligned}$$

We have $p(z, M, m|G, \Lambda)$, from Bayes theorem, In Catalog as

$$\begin{aligned} p(z, M, m|G, \Lambda) &= \frac{p(z, M, m|G, \Lambda \setminus \{s\}) p(s|z, M, m, G, \Lambda)}{p(s|G, \theta', \Lambda)} \\ &= \frac{1}{p(s|G, \theta', \Lambda)} \delta(M - \mathcal{M}(z, m, \Lambda)) p(z, m|G, \theta') p(s|z, M, \Lambda) \end{aligned}$$

Thereby, the integrals,

$$\int \int dm dM p(z, M, m|G, \Lambda) = \frac{1}{p(s|G, \theta', \Lambda)} \int dm p(z, m|G, \theta') p(s|z, M(z, m, \Lambda), \Lambda)$$

where we have the incorporation of redshift uncertainty into the galaxies in the likelihood, as

$$p(z, m|G, \theta') = \frac{1}{N_{\text{gal}}(\theta')} \sum_k^{N_{\text{gal}}(\theta')} p(z|\hat{z}_k) \delta(z - \hat{z}_k)$$

Similarly, for the Out Catalog term, we have a purely cosmological dependent term, we use the selection threshold such that,

$$p(\tilde{G}|z, M, m, \theta', H_0) = \Theta[m - m_{\text{thr}}(\theta)] \Theta[z_{\text{cut}} - z] + \Theta[z - z_{\text{cut}}]$$

where the galaxies fainter than magnitude threshold but within redshift range, and galaxies outside the range. The integral simplifies as,

$$\begin{aligned} \int \int dm dM p(z, M, m|\tilde{G}, \theta', \Lambda, s) = & \frac{1}{p(s|\tilde{G}, \theta', \Lambda) p(\tilde{G}|z, M, m)} \left\{ \Theta[z_{\text{cut}} - z] \int_{M(z, m_{\text{th}}, \Lambda)}^{M_{\text{max}}} dM p(M|\Lambda) p(z|M) \right. \\ & \left. + \Theta[z - z_{\text{cut}}] \int_{M_{\text{min}}}^{M_{\text{max}}} dM p(M|\Lambda) p(z|M) \right\} \end{aligned}$$

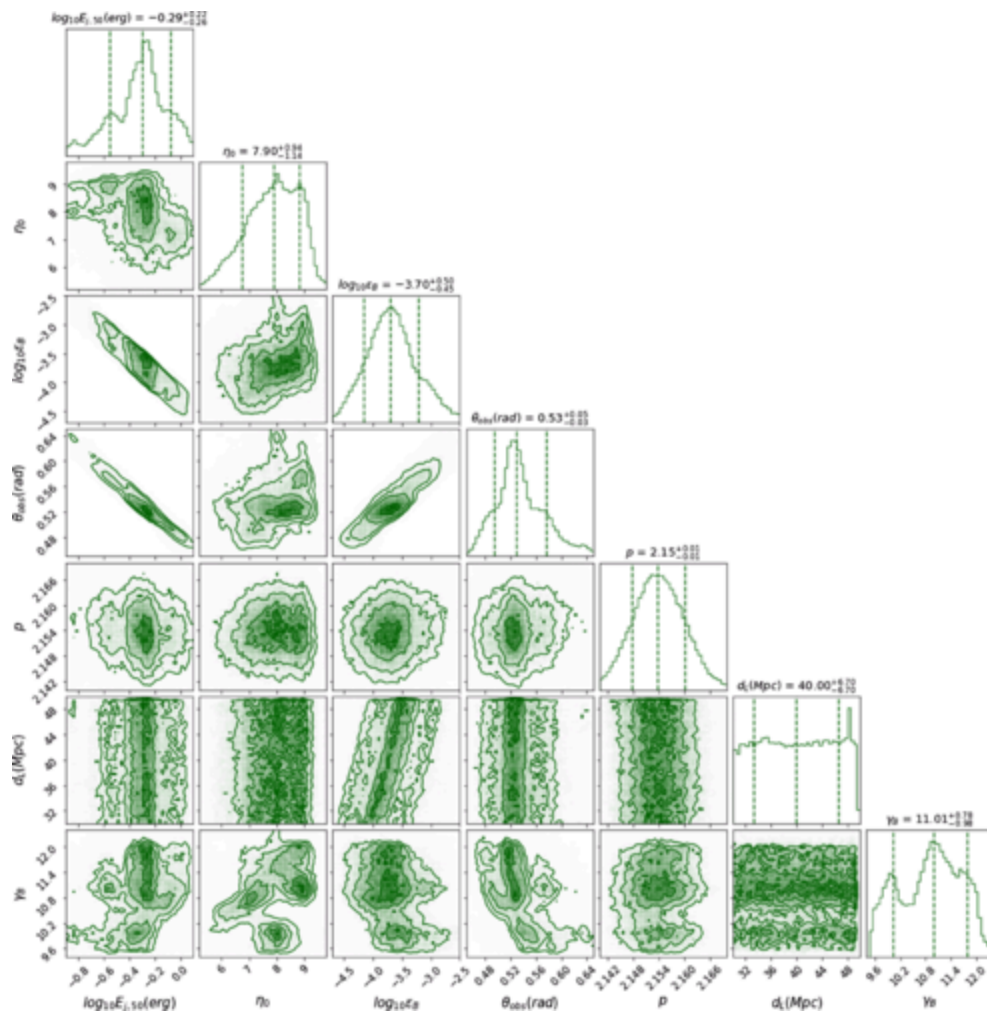
For the redshift line of sight prior, we can also separate parameters, *cosmological*, or correspond to normalization or shape or mass distribution,

$$p(s|z, M, \Lambda) = p(s|z, M, \Lambda_{\text{rate}}) p(s|z, \Lambda_{\text{rate}}) p(s|M)$$

where we can have uniform or luminosity weighted sampling for $p(s|M)$. The line of sight prior can be pre-computed in offline analyses.

Sampled Posteriors

Degeneracy between Hubble constant H_0 and features of mass distribution upto the maximum mass of the black hole.



L01.02 GW Probes for the Early Universe

Introduction

Weakness of gravitational waves interaction, universe is *transparent* to GWs,

$$\frac{\Gamma(T)}{R(T)} \sim \frac{G^2 T^5}{T^2/M_{\text{Planck}}} \sim \left(\frac{T}{M_{\text{Planck}}} \right) < 1$$

decoupling of the gravitational waves, due to less effective cross section rate. The interaction rate $\Gamma(T)$ is assuming weak interaction at temperature T . Similar calculations for photons, neutrinos show interaction.

Similar analogy to the Cosmic Background. *Stochastic GW Backgrounds*, a fossil radiation, of the earlier universe, at those energy scales of GW wave generation.

Flat Space Gravitational Field Background

First-order perturbation $|h_{\mu\nu}(\mathbf{x})| \ll 1$, with metric

$$g_{\mu\nu}(\mathbf{x}) = \eta_{\mu\nu} + h_{\mu\nu}(\mathbf{x})$$

Linearise in $h_{\mu\nu}(\mathbf{x})$, with metric $g_{\mu\nu}(\mathbf{x}) \sim \eta_{\mu\nu}$ and its inverse $g^{\mu\nu}(\mathbf{x}) \sim \eta^{\mu\nu}(\mathbf{x})$ in the first order for contractions. Affine connection,

$$\Gamma_{\mu\nu}^{\alpha} \simeq \frac{1}{2} (\partial_{\nu} h_{\mu}^{\alpha} + \partial_{\mu} h_{\nu}^{\alpha} - \partial^{\alpha} h_{\mu\nu})$$

Riemann tensor, being *invariant* under coordinate transformations,

$$R_{\mu\nu\beta}^{\alpha} \simeq \frac{1}{2} (\partial_{\mu} \partial_{\nu} h_{\beta}^{\alpha} + \partial_{\beta} \partial^{\alpha} h_{\mu\nu} + \partial_{\nu} \partial^{\alpha} h_{\mu\beta} - \partial_{\beta} \partial_{\mu} h_{\nu}^{\alpha})$$

Einstein tensor,

$$G_{\mu\nu} = \frac{1}{2} (\partial_{\alpha} \partial_{\nu} \tilde{h}_{\mu}^{\alpha} + \partial^{\alpha} \partial_{\mu} \tilde{h}_{\nu\alpha} - \square \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \partial_{\alpha} \partial^{\beta} \tilde{h}_{\beta}^{\alpha})$$

where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$ is the trace free stress tensor.

Gauge Transformations

Since General relativity is invariant under coordinate transformations, we have linearised perturbed GR being invariant under slowly varying coordinate transformations.

We have $x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \xi^{\mu}$, thereby, $h_{\mu\nu}(\mathbf{x}) \rightarrow h'_{\mu\nu}(\mathbf{x}) = h_{\mu\nu}(\mathbf{x}) - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}$, invariance maintained under $|\partial_{\alpha} \xi_{\beta}| \ll |h_{\alpha\beta}|$ such that $||h'_{\mu\nu}|| \ll 1$.

Lorenz Transformations

Consider $x^\mu \rightarrow x'^\mu = a^\mu + \Lambda^\mu_\nu x^\nu + \xi^\nu$, where a^μ are some constants, Λ^μ_ν is the Lorenz transformation, and ξ^μ is a small gauge transformation. Inverting, we have $x^\gamma = \Lambda^\gamma_\mu x'^\mu - \Lambda^\gamma_\mu - \Lambda^\gamma_\mu \xi^\mu$, thereby, the derivative $\frac{\partial x^\gamma}{\partial x^\alpha} \simeq \Lambda^\gamma_\alpha - \Lambda^\gamma_\mu \frac{\partial \xi^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\alpha}$ in the first order. Thereby, we have the metric

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} \simeq \eta_{\alpha\beta} + \underbrace{[h'_{\alpha\beta} - \Lambda^\sigma_\beta \xi_{\alpha\sigma} - \Lambda^\sigma_\alpha \xi_{\sigma\beta}]}_{=h''_{\alpha\beta}}$$

where $h'_{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta h_{\mu\nu}$. Propagation on the flat metric, as an effective background. Thereby, $h''_{\mu\nu}$ can be treated as a 2-tensor on a flat background under Lorenz gauge transformations.

The derivative transforms as $\partial^\mu \tilde{h}_{\mu\nu}(\mathbf{x}) \rightarrow \partial'^\mu \tilde{h}'_{\mu\nu}(\mathbf{x}) = \partial^\mu \tilde{h}_{\mu\nu}(\mathbf{x}) - \square \xi_\nu$, Lorenz Gauge sets $\partial^\mu \tilde{h}'_{\mu\nu}(\mathbf{x}) = 0$, which can be done by solving the equation $\partial^\mu \tilde{h}_{\mu\nu}(\mathbf{x}) = \square \xi_\nu$. The Poisson equation is simplified as $\square \tilde{h}_{\mu\nu} = \partial^\mu \tilde{h}_{\mu\nu} = \mathcal{J}_\nu$, in flat spacetime, as the Dirichlet boundary conditions are met. This simplifies the derivatives in the Einstein tensor, by setting it to zero.

From the Lorenz gauge, we have the Einstein equations being represented as a Wave equation,

$$\square \tilde{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Thereby, the gauge $\partial^\mu \tilde{h}_{\mu\nu}(\mathbf{x}) = 0$ implies $\partial^\mu T_{\mu\nu} = 0$. Implies energy-momentum tensor of the source is *conserved*. Thereby, the source does not lose energy and momentum by emission in linearised theory, source described by Newtonian gravity. Linearised theory does not describe how GW emission influences the source, but describes behavior of test masses.

We have symmetries $\tilde{h}_{\mu\nu} = \tilde{h}_{\nu\mu}$ implying 10 components, with further the Lorenz gauge, $\partial^\mu \tilde{h}_{\mu\nu}(\mathbf{x}) = 0$ reducing to 4 independent components. These are not all physical gauge transformations, residual gauge freedom exists in the Lorenz gauge.

Due to residual degrees of freedom, performing the coordinate transformation under the Lorenz gauge, with $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu$, with $\square \xi^\mu = 0$, to remain in the gauge. We have the relations, $h_{\mu\nu}(\mathbf{x}) \rightarrow h'_{\mu\nu}(\mathbf{x}) = h_{\mu\nu}(\mathbf{x}) - \xi_{\mu\nu}$ where $\xi_{\mu\nu} = \eta_{\mu\nu} \partial^\alpha \xi_\alpha - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. We see the generalized conditions, $\square \xi_{\mu\nu} = 0$ since the commuting derivatives in vacuum, and $\square \xi_\mu = 0$. Further, we have $\square \tilde{h}_{\mu\nu} \rightarrow \square' \tilde{h}'_{\mu\nu} \simeq \square(\tilde{h}_{\mu\nu} + \xi_{\mu\nu})$. By the additional gauge freedom, we set $\tilde{h} = 0$ and hence $h_{\mu\nu} = \tilde{h}_{\mu\nu}$, and further $\tilde{h}_{i0} = 0$. Further, by our earlier Lorenz gauge, $\partial^\mu h_{\mu 0} = 0$, and hence $\nabla^2 h_{00} = 0$, and thereby, we arrive at the further constraints for $h_{\mu\nu}$. We have the 8 constraints, $h_{0\mu} = 0$, $h^\mu_\mu = 0$ and $\partial^i h_i = 0$.

Restricting to vacuum spacetime, the residual coordinate freedom can be used to fix the constraints, **Transverse-Traceless Gauge***. Traceless: $\tilde{h}'^\mu_\mu = 0$ and Transverse $h'_{0i} = 0$ with

further conditions $\partial^i h'_{ij} = 0$ and $h_{00} = 0$. Since $\square \tilde{h}_{\mu\nu} = 0 \rightarrow \square \tilde{h} = 0$. The trace transforms as $\tilde{h} \rightarrow \tilde{h}' = \tilde{h} - 2\partial^\mu \xi_\mu$ thereby, $\square \tilde{h} = 0$ and therefore, only because of vacuum.

Thereby, the number of independent components in n dimensions is $\frac{n(n+1)}{2} - (n + n) = \frac{n(n-3)}{2}$, where the $\frac{n(n+1)}{2}$ symmetric components, are constrained by n gauge conditions, and n residual degrees of freedom. For $n = 4$, we have 2 physical degrees of freedom.

GW Waves

Thereby, only 2 remaining physical degrees of freedom in the metric leading to

$$\square h_{ij}(\mathbf{x}') = 0$$

since we can chose 4 functions ξ_μ freely. These two independent polarisations are summed as

$$h_{ij}(\mathbf{x}) = \sum_{r=+, \times} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} h_r(\mathbf{k}) e_{ij}^r(\hat{\mathbf{k}}) \exp(-ik(t - \mathbf{k} \cdot \mathbf{x}))$$

which are plane waves, transverse, moving at the speed of light. The polarisation tensors are transverse in $[m, n]$ plane

$$e_{ij}^+(\hat{\mathbf{k}}) = \hat{m}_i \hat{m}_j - \hat{n}_i \hat{n}_j$$

$$e_{ij}^\times(\hat{\mathbf{k}}) = \hat{m}_i \hat{n}_j + \hat{n}_i \hat{m}_j$$

where the free waves are assumed to travel in the z direction. We have

$$h_{ij}(z, t) = \begin{pmatrix} h_+ & h_\times & 0 \\ -h_\times & h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos(\omega(t - z))$$

Metric line element is thereby,

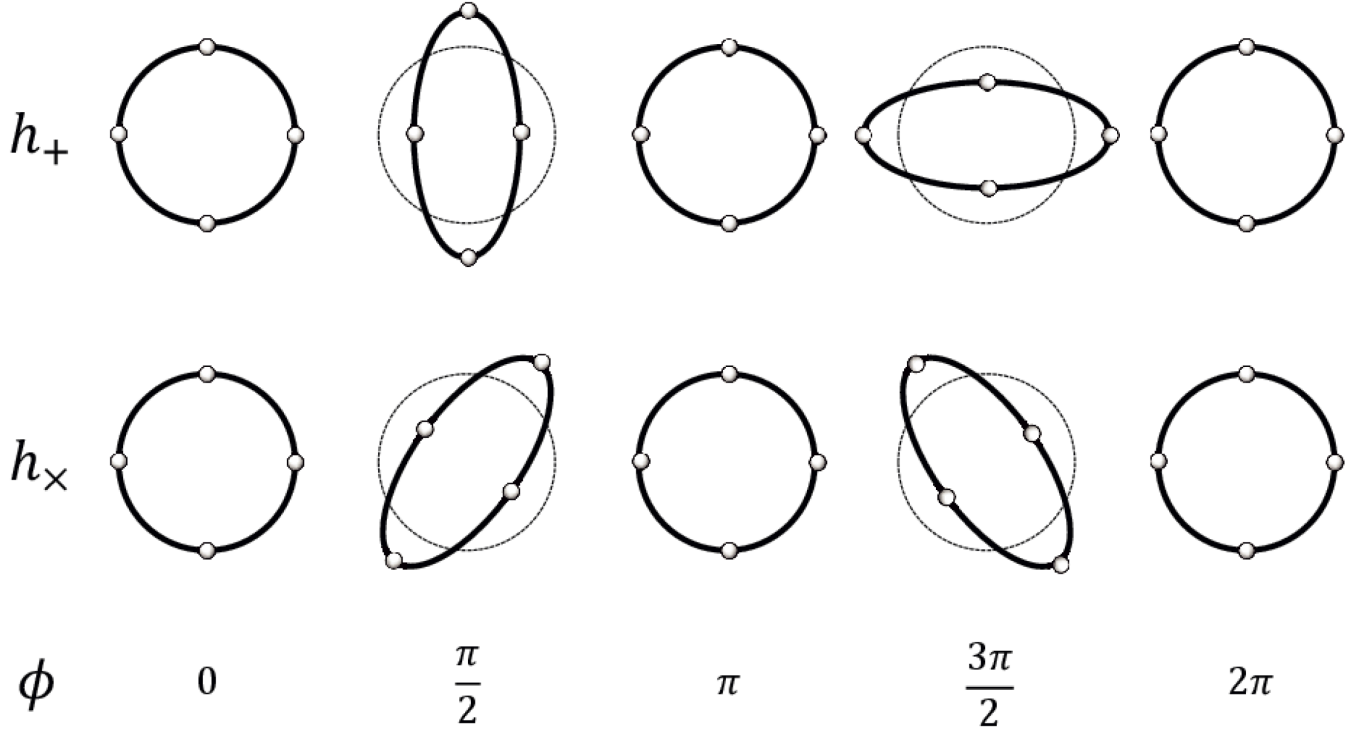
$$ds^2 = -dt^2 + dz^2 + (1 + h_+ \cos(\omega(t - z)))dx^2 + (1 - h_\times \cos(\omega(t - z)))dy^2$$

Action on Test masses

Consider plane propagating waves in $\hat{r}$ direction, $\tilde{h}_{\mu\nu} \sim \mathcal{A}_{\mu\nu} \exp(ik_r \cdot \hat{r})$, thereby, $\square \tilde{h}_{\mu\nu} = -\tilde{h}_{\mu\nu} \eta^{\alpha\beta} k_\alpha k_\beta = 0$, implying that, $k^\alpha k_\alpha = 0$, thereby, $\omega^2 = k_r^2$. Further, we use the Lorenz gauge, to have $\partial^\mu \tilde{h}_{\mu\nu} = 0$, thereby $\mathcal{A}_{\mu\nu} k^\nu = 0$, hence $\mathcal{A}_{\mu 0} = \mathcal{A}_{\mu r}$, which reduces to 6 independent components. Further, with the additional gauge freedom, we can relate the in the new coordinates, $\tilde{h}_{\mu\nu} \rightarrow \tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial^\alpha \xi_\alpha$. Thereby, we have $\mathcal{A}_{\mu\nu} \rightarrow \mathcal{A}'_{\mu\nu} = \mathcal{A}_{\mu\nu} - ik_\mu \xi_\nu - ik_\nu \xi_\mu + \eta_{\mu\nu} ik_\alpha \xi^\alpha$. This reduces further constraints to only 2 independent quantities of $\mathcal{A}_{\mu\nu}$.

Geodesic deviation equation $\xi^i = -R^i_{j0} \xi^j = \frac{1}{2} \ddot{\tilde{h}}_{ij} \xi^j$ in the TT gauge. Thereby, the evolution is directed by the second derivative of the metric tensor perturbation. The action on test masses

can be seen through the effective displacements.



Quantisation

Polarisation is related to the spin of massless particle expected upon quantisation $S = \frac{2\pi}{\theta}$, where θ is the invariance angle of generic rotation. On transformation by $\theta = 2\pi$ results in spin-2 quantisation which are the two independent degrees of freedom which are physical. GWs have only two physical components is a manifestation of the intrinsic nature of gravitational interaction, mediated by the *graviton*, a spin-2 massless field, that has only two independent helicity states. Thereby, true with spacetime with matter fields.

Matter Field Gravitational Background

Metric

$$g_{\mu\nu}(\mathbf{x}) = \eta_{\mu\nu} + h_{\mu\nu}(\mathbf{x})$$

with splitting into irreducible components under rotation.

$$\begin{aligned} h_{00} &= \underbrace{-2\phi}_{\text{Scalar Trace}} \\ h_{0i} &= \underbrace{\partial_i B}_{\text{Scalar}} + \underbrace{S_i}_{\text{Divergence-Free Vector}} \quad (\partial_i S_i = 0) \\ h_{ij} &= \underbrace{-2\psi\delta_{ij}}_{\text{Scalar Trace}} + \underbrace{\left(\partial_i\partial_j - \frac{1}{3}\nabla^2\right)}_{\text{Scalar}} + \underbrace{\partial_j F_i}_{\text{Vector}} + \underbrace{H_{ij}}_{\text{Tensor}} \end{aligned}$$

Now, infinitesimal coordinate transformations, $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu$,

$h_{\mu\nu}(\mathbf{x}) \rightarrow h'_{\mu\nu}(\mathbf{x}) = h_{\mu\nu}(\mathbf{x}) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$, with the decomposition, $\xi^\mu = (\xi_0, \xi_i) = (d_0, \partial_i d + d_i)$, thereby we have the invariant coordinate transformations,

$$\begin{aligned}\Phi &= \phi + \dot{B} - \frac{1}{2}\ddot{E} \quad \text{Scalar} \\ \Theta &= -2\psi - \frac{1}{3}\nabla^2 E \quad \text{Scalar} \\ \Sigma_i &= S_i - \dot{F}_i \quad (\partial_i \Sigma_i = 0) \quad \text{Vector} \\ \mathcal{H}_{ij} &= H_{ij} \quad (\partial_i \mathcal{H}_{ij} = 0, \mathcal{H}_i^i = 0) \quad \text{Tensor}\end{aligned}$$

Two scalars, one vector and one tensor gauge invariant variables. Six independent degrees of freedom of the metric.

For the $T_{\mu\nu}$, we have the components,

$$\begin{aligned}T_{00} &= \underbrace{\rho}_{\text{Scalar}} \\ T_{0i} &= \underbrace{\partial_i u}_{\text{Scalar}} + \underbrace{u_i}_{\text{Divergence-Free Vector}} \quad (\partial_i u_i = 0) \\ T_{ij} &= \underbrace{p\delta_{ij}}_{\text{Scalar Trace}} + \underbrace{\left(\partial_i \partial_j - \frac{1}{3}\nabla^2\right)\sigma}_{\text{Scalar}} + \underbrace{\partial_i v_j + \partial_j v_i}_{\text{Vector}} + \underbrace{\Pi_{ij}}_{\text{Tensor}}\end{aligned}$$

Similar to before, 4 independent quantities, due to energy-momentum conservation.

Rewriting the Einstein equation in terms of the 6 gauge invariant variables,

$$\begin{aligned}\nabla^2 \Theta &= -8\pi G \rho \\ \nabla^2 \Phi &= 4\pi G (\rho + 3p - 3\dot{u}) \\ \nabla^2 \Sigma_i &= -16\pi G S_i \\ \square \mathcal{H}_{ij} &= -16\pi G \Pi_{ij}\end{aligned}$$

Three Poisson-like-equations, one wave equation. Only the TT metric components are radiative. Only the TT metric components are radiative.

L02.02 Cosmological Gravitational Wave Background



Gravitational waves only change the proper distance between the masses, and not their coordinate distances.

For a particle at rest initially, with world line $x^\mu(t)$, we have

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

which can be simplified in the TT gauge to arrive at $\frac{d^2 x^\mu}{d\tau^2} = 0$, implying x^μ must remain constant, since it is at rest initially.

However, the proper distance defined through the metric

$$\ell = \int ds = \int \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}} d\tau$$

is shown to change with the propagation of GW waves.

The proper distance is related to the coordinate distance with the effective scale due to the GW wave, and the coordinate distance is constant.

GW Energy-Momentum Tensor and GW Propagation

Background spacetime has maximally symmetric sub-manifolds. We study the propagation and generation of a background of fluctuation scale L_B . Resorting to a clear separation of scales.

Relative scale of spatial variations are small, $|h_{\mu\nu}| \ll 1$ and $\frac{\lambda}{L_B} \ll 1$ and $\frac{f_B}{f_L} \ll 1$.

Distinction between background and GWs, *averaging* of physical quantities, $\lambda \ll \bar{\ell} \ll L_B$ as the characteristic scale, $f_B \ll \bar{f} \ll f$, we average the full metric over the length scales, removing high frequency or smaller lengths, such that $\langle g_{\mu\nu} \rangle = \bar{g}_{\mu\nu}$, and $\langle h_{\mu\nu} \rangle = 0$.

Expanding the Einstein equations to second order in $h_{\mu\nu}$ (since linearisation gives zeros), through $R_{\mu\nu} = \bar{R}_{\mu\nu} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)}$, where the quadratic term can influence the background, averaging at second order.

Background Equation

Background Einstein equation, with $\langle \dots \rangle = [\dots]^{\text{low}}$, such that,

$$\bar{R}_{\mu\nu} = \left[-R_{\mu\nu}^{(2)} \right]^{\text{low}} + 8\pi G \left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right]^{\text{low}}$$

where $\bar{R}_{\mu\nu}$ is the background curvature in $O((1/L_B)^2)$, $\left[-R_{\mu\nu}^{(2)} \right]^{\text{low}}$ sources the curvature of the background in $O((h/\lambda)^2)$, and $T_{\mu\nu}$ is the matter density only sourcing the background curvature. By comparing the orders, we have the perturbation,

$$h \lesssim \frac{\lambda}{L_B}$$

Rearranging, and performing the average,

$$\bar{G}_{\mu\nu} = \langle R_{\mu\nu} \rangle - \frac{1}{2} \bar{g}_{\mu\nu} \langle R \rangle = 8\pi G (\langle T_{\mu\nu} \rangle + T_{\mu\nu}^{\text{GW}})$$

where $T_{\mu\nu}^{\text{GW}}$ is the gravitational wave energy-momentum tensor, given by

$$T_{\mu\nu}^{\text{GW}} = -\frac{1}{8\pi G} \left\langle R_{\mu\nu}^{(2)} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \right\rangle$$

and reducing in TT gauge, we have,

$$T_{\mu\nu}^{\text{GW}} = \frac{1}{32\pi G} \langle \nabla_\mu h_{\alpha\beta} \nabla_\nu h^{\alpha\beta} \rangle$$

where we note that the Bianchi identity, is for the entire energy momentum conservation, including the gravitational wave emission. The energy density of GW can be derived from the trace, as,

$$\varrho_{\text{GW}} = \langle \dot{h}_{ij} \dot{h}^{ij} \rangle$$

which is gauge-independent.

Perturbed Einstein equation

Now, we focus on the high modes, given by the linear term,

$$R_{\mu\nu}^{(1)} = \left[-R_{\mu\nu}^{(2)} \right]^{\text{high}} + 8\pi G \left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right]^{\text{high}}$$

where here the $\left[-R_{\mu\nu}^{(2)} \right]^{\text{high}}$ is negligible as the non-linear interaction of the wave, and $T_{\mu\nu}$ can be the source terms. The perturbed Einstein equation, is thereby,

$$R_{\mu\nu}^{(1)} - \frac{1}{2} (\bar{g}_{\mu\nu} R^{(1)} + h_{\mu\nu} \bar{R}) \simeq 8\pi G [T_{\mu\nu}]^{\text{high}}$$

where the evolution of GWs on the curved, but smooth/slowly evolving background is projected. The higher $T_{\mu\nu}$ contributes as a possible source of GWs.

Thereby, we can calculate by expanding the Riemann tensor,

$$-\frac{1}{2}\square\tilde{h}_{\mu\nu} + R_{\mu\nu\sigma}^{\lambda}\tilde{h}_{\lambda}^{\sigma} + \nabla_{(\nu}\nabla^{\sigma}\tilde{h}_{\mu)\sigma} - \frac{1}{2}\bar{g}_{\mu\nu}\nabla^{\alpha}\nabla^{\beta}\tilde{h}_{\alpha\beta} + R^{\alpha\beta}\left[\frac{1}{2}\bar{g}_{\mu\nu}\tilde{h}_{\alpha\beta} - \frac{1}{2}\tilde{h}_{\mu\nu}\bar{g}_{\alpha\beta} + \bar{g}_{\beta(\mu}\tilde{h}_{\nu)\alpha}\right] = 8\pi G\delta T_{\mu}$$

Propagation in FLRW

For the FLRW metric, we have, in the gauge, $\partial_i h_{ij} = 0$, and $h_i^i = 0$,

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j$$

with solving the above equations in the traceless tensor,

$$\ddot{h}_{ij}(\mathbf{x}) + 3H\dot{h}_{ij}(\mathbf{x}) - \frac{\nabla^2}{a^2(t)}h_{ij}(\mathbf{x}) = 16\pi G\Pi_{ij}(\mathbf{x})$$

The source tensor represents the anisotropic stress, with no gravitational wave from the homogeneous component.

Using the translational invariance, we perform a Fourier transform in space, through

$$h_{ij}(\mathbf{x}) = \sum_{r=+, \times} \int \frac{d^3\mathbf{k}}{(2\pi)^3} h_r(\mathbf{k}, t) e_{ij}^r(\hat{\mathbf{k}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

where the time dependence is still present in $h_r(\mathbf{k}, t)$, unlike Minkowski. Similarly, we can decompose the source as,

$$\Pi_{ij}(\mathbf{x}) = \sum_{r=+, \times} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Pi_r(\mathbf{k}, t) e_{ij}^r(\hat{\mathbf{k}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

where we can decouple for each polarisation mode, as

$$h_r''(\mathbf{k}, \tau) + 2\mathcal{H}h_r'(\mathbf{k}, \tau) + k^2 h_r(\mathbf{k}, \tau) = 16\pi G a^2(\tau) \Pi_r(\mathbf{k}, \tau)$$

where we use the conformal time derivatives.

For the homogeneous equation without source, we can have the power law scale factor $a(\tau) = a_n \tau^n$, and we can solve in terms of Bessel functions, as

$$h_r(\mathbf{k}, \tau) = \frac{A_r(\mathbf{k})}{a_n \tau^{n-1}} j_{n-1}(k\tau) + \frac{B_r(\mathbf{k})}{a_n \tau^{n-1}} y_{n-1}(k\tau)$$

where can transform $H_r(\mathbf{k}, \tau) = a(\tau)h_r(\mathbf{k}, \tau)$, where $H_r''(\mathbf{k}, \tau) + \left(k^2 - \frac{a''}{a}\right)H_r(\mathbf{k}, \tau) = 0$ when $\frac{a''}{a} \propto \mathcal{H}$, with two limiting cases: *sub-Hubble modes*, for $k^2 \gg \mathcal{H}^2$ where we have plane GW

waves with redshifted amplitude, and *super-Hubble modes*, where $k^2 \ll h^2$ and relevant solution under inflationary initial conditions.

We have defined GWs and GW energy density without ambiguity in the FLRW spacetime, which oscillate and decay with the expansion of the universe.

Stochastic GW Background

Only statistical properties can be accessed, due to incoherent superposition of sources which cannot be individually resolved. Confusion noise, indeterministic combination of deterministic sources.

For a GW source at time t_* in the early universe, which cannot produce a signal on length/time scales larger than the causal horizon, with

$$\ell_* \leq H_*^{-1}$$

where the characteristic length scale of the source is ℓ_* , representing the size of variation of the tensor anisotropic stresses.

The angular size on the sky today of a region in which the SGWB signal is correlated, is

$$\Theta_* = \frac{\ell}{d_A(z_*)}$$

such that the number of uncorrelated regions accessible currently is around Θ_*^{-2} , hence order of magnitude analysis, results that the details of the signal cannot be accessed.

We can reduce the ensemble average to the volume/time average by the ***Ergodic Hypothesis***, with different realizations of the GW signal in the homogeneous and the isotropic nature of the universe.

L03.02 Stochastic Gravitational Wave Backgrounds

Primordial SGWB

The primordial SGWBs are homogeneous (similar to the FLRW spacetime, with the correlators depending on the relative proper distance, $\langle h_{ij}(\mathbf{x}, \tau_1) h_{kl}(\mathbf{y}, \tau_2) \rangle = \mathcal{F}_{ijkl}(|\mathbf{x} - \mathbf{y}|, \tau_1, \tau_2)$) and isotropic (there exists induced anisotropy, like the dipole with respect to the cosmological frame, more challenging than the monopole), unpolarised and Gaussian (*Central Limit Theorem*, independent random realisations).

Power spectrum of the GW amplitude, using the cross correlation

$$\langle h_r(\mathbf{k}, \tau) h_p^*(\mathbf{q}, \tau) \rangle = \frac{8\pi^5}{k^3} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{rp} h_c^2(k, \tau)$$

which entails, the characteristic nature similar to the CMB. It is the second moment, as the two-point correlation function.

For the GW energy density ϱ_{GW} , we note,

$$\varrho_{\text{GW}} = \frac{\langle \dot{h}_{ij}(\mathbf{x}, t) \dot{h}_{ij}(\mathbf{x}, t) \rangle}{32\pi G} = \frac{\langle \dot{h}_{ij}(\mathbf{x}, \tau) \dot{h}_{ij}(\mathbf{x}, \tau) \rangle}{32\pi G a^2(\tau)} = \int_0^\infty \frac{dk}{k} \frac{\partial \varrho_{\text{GW}}}{\partial \ln k}$$

For free propagating sub-Hubble modes, we can expand as plane waves in the Fourier space, using the freely propagating plane wave, and time-average by approximating $h_c'^2(k, \tau) \sim k^2 h_c^2(k, \tau)$. We have the same structure for the cross correlation function as above, by using quasi-static nature of the spacetime, such that we can Fourier transform, such that,

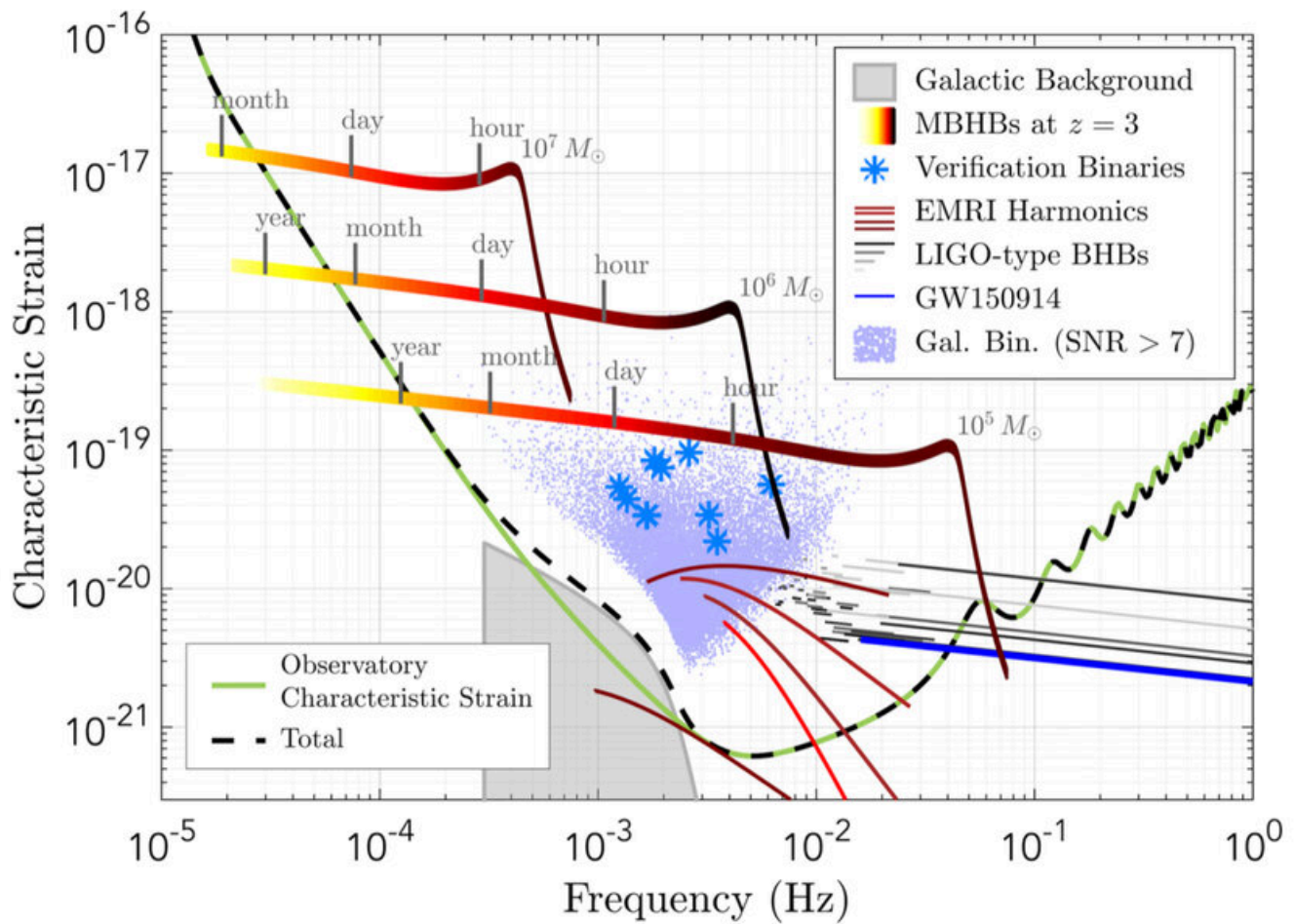
$$\langle h_r'(\mathbf{k}, \tau) h_p'^*(\mathbf{q}, \tau) \rangle = \frac{8\pi^5}{k^3} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{rp} h_c'^2(k, \tau)$$

The power spectrum of the GW energy density can be related to the power spectrum of the GW amplitude as

$$\frac{\partial \varrho_{\text{GW}}}{\partial \ln k} = \frac{k^2 h_c^2(\tau)}{16\pi G a^2(\tau)}$$

Prospectus: LISA

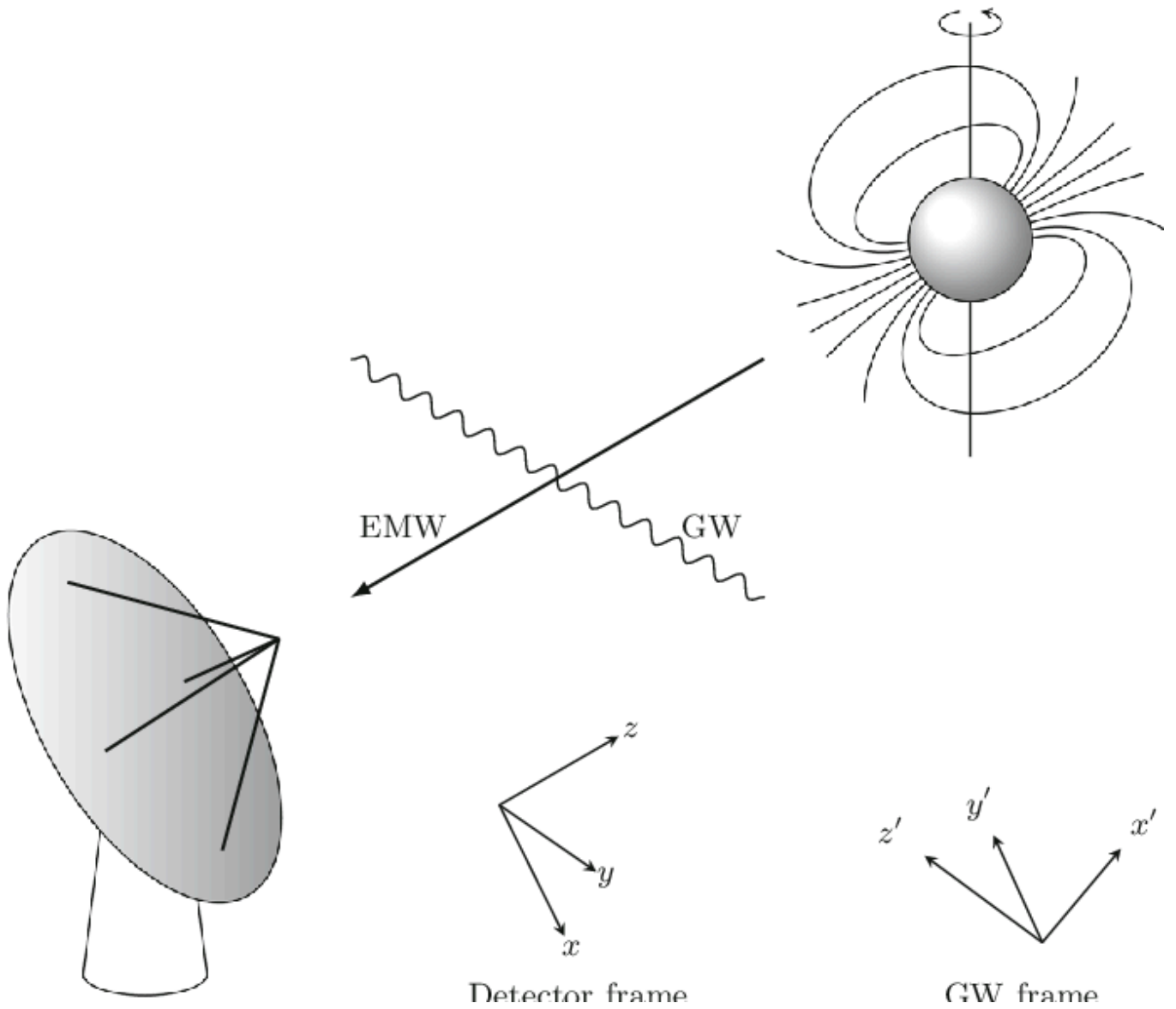
Confusion noise detection from binaries, and stochastic GW background.



Null channel evidence is required, due to lack of *cross-correlation*.

Pulsar Timing Array

Rotating, magnetized neutron stars emitting periodic radio-frequency EM pulses, whose arrival times can be modeled by *timing residuals*, studying effects of passing by gravitational wave.



Through gravitational redshift, caused by waves emitted by far-away sources, and travelling through spacetime between the pulsars and the earth. Photon from the pulsar, received at the arrival time

$$dt = \pm \sqrt{(\delta_{ij} + h_{ij})dx^i dx^j}$$

which leads us to,

$$t_o - t_e = L + \frac{1}{2} \hat{\mathbf{u}}^i \hat{\mathbf{u}}^j \int_0^L ds h_{ij}(t_e + s, \mathbf{x} + s\hat{\mathbf{u}})$$

Effect of metric perturbations on a single beam.

Principle of measurement, is to compare alternate pulses, thereby, the time residual shift, is given by

$$\Delta T = \frac{1}{2} \hat{\mathbf{u}}^i \hat{\mathbf{u}}^j \int_0^L ds \left[h_{ij}(t_e + s + \mathcal{P}, \mathbf{x} + s\hat{\mathbf{u}}) - h_{ij}(t_e + s, \mathbf{x} + s\hat{\mathbf{u}}) \right]$$

due to the varying time-dependence of gravitational wave.

Thereby, we have

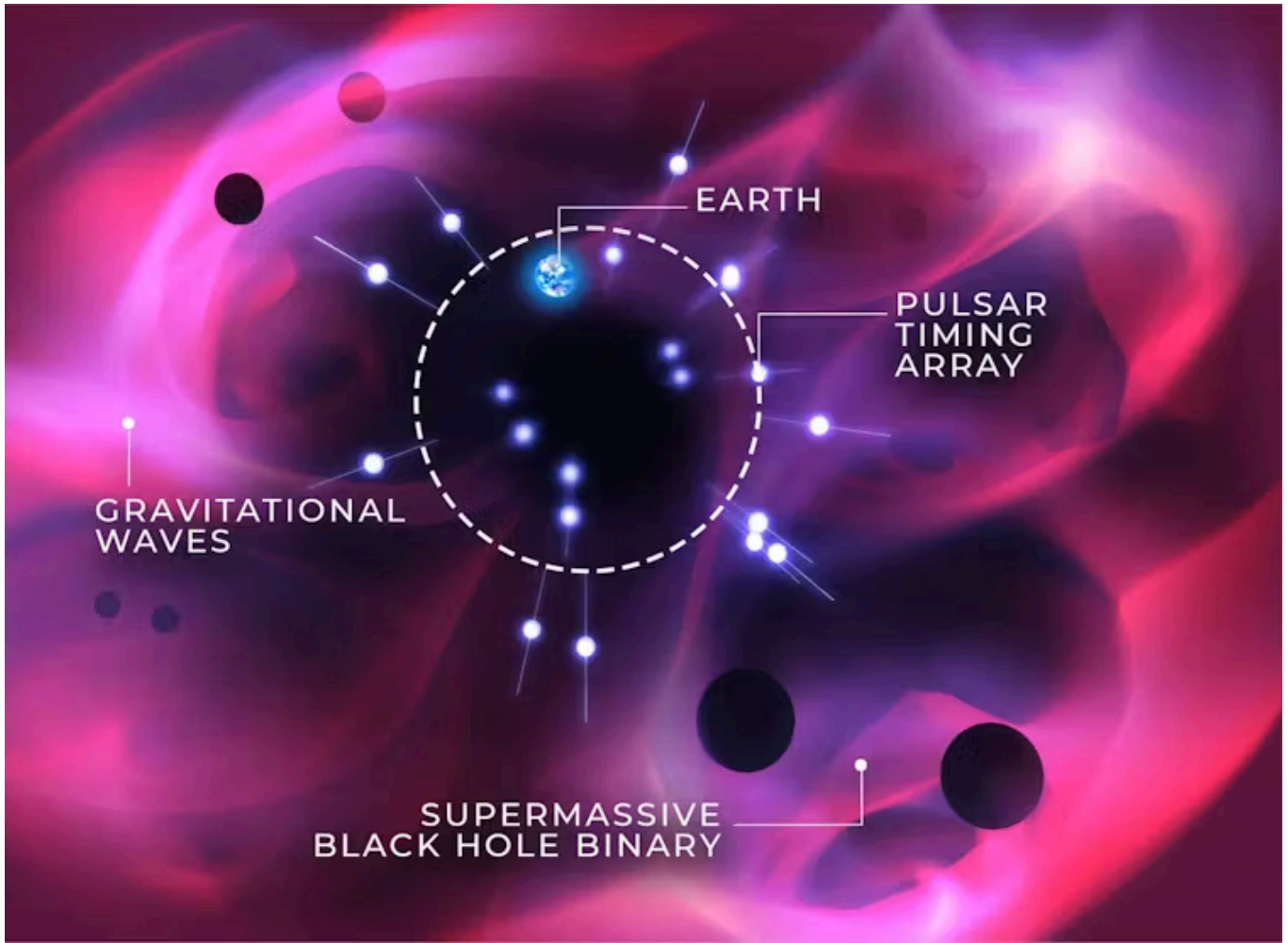
$$\Delta T = \frac{1}{2} \frac{\hat{\mathbf{u}}^i \hat{\mathbf{u}}^j}{1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}}} \int_{t_e - \hat{\mathbf{k}} \cdot \mathbf{x}}^{t_e - \hat{\mathbf{k}} \cdot \mathbf{x} + L(1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}})} ds \left[h_{ij}(X + \mathcal{P}) - h_{ij}(X) \right]$$

which can be Taylor expanded, as

$$\frac{\Delta T}{\mathcal{P}} \simeq \frac{1}{2} \frac{\hat{\mathbf{u}}^i \hat{\mathbf{u}}^j}{1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}}} \left[\underbrace{h_{ij}(t_e + L, \mathbf{x}_0)}_{\text{Earth Term}} - \underbrace{h_{ij}(t_e, \mathbf{x}_0)}_{\text{Pulsar Term}} \right]$$

where we can estimate the scale of time variation, with a semi-classical approach.

L04.02 Pulsar Timing Arrays



Time Delay Analysis

We have

$$\Delta T = \frac{1}{2} \frac{\hat{\mathbf{u}}^i \hat{\mathbf{u}}^j}{1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}}} \int_{t_e - \hat{\mathbf{k}} \cdot \mathbf{x}}^{t_e - \hat{\mathbf{k}} \cdot \mathbf{x} + L(1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}})} ds [h_{ij}(X + \mathcal{P}) - h_{ij}(X)]$$

Noting the frequency evolution (chirp) as

$$f_{\text{GW}}(\tau) = \frac{1}{\pi} \left(\frac{5}{256} \right)^{\frac{3}{8}} \frac{1}{(GM_c)^{\frac{5}{8}} \tau^{\frac{3}{8}}}$$

where we understand that $f_{\text{GW}} \mathcal{P} \ll 1$, thereby, h_{ij} is Taylor expanded, as

$$\frac{\Delta T}{\mathcal{P}} \simeq \frac{1}{2} \frac{\hat{\mathbf{u}}^i \hat{\mathbf{u}}^j}{1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}}} \underbrace{[h_{ij}(t_e + L, \mathbf{x}_0)]}_{\text{Earth Term}} - \underbrace{[h_{ij}(t_e, \mathbf{x}_0)]}_{\text{Pulsar Term}}$$

The GW wave at earth, and at the pulsar differ by a characteristic time delay.

The timing residuals are estimated as

$$R(T) = \int_{t_{\text{ref}}}^{t_{\text{ref}}+T} dt \frac{\Delta T}{\mathcal{P}}$$

Measurement plausible through extremely small residual order, with the precision of pulsar monitoring.

Correlations between many pulsars to reduce noise. For the earth term, we have $t_e + L - |\mathbf{x}_0 - \mathbf{x}_s|$, which dominates since the pulsar term varies. The cross correlation is related as

$$\langle R_a(T) R_b(T) \rangle = \int_{t_{\text{ref}}}^{t_{\text{ref}}+T} dt' \int_{t_{\text{ref}}}^{t_{\text{ref}}+T} dt'' \left\langle \frac{\Delta T_a}{\mathcal{P}_a}(t') \frac{\Delta T_b}{\mathcal{P}_b}(t'') \right\rangle$$

where we expand

$$\frac{\Delta T_i}{\mathcal{P}_i} = \sum_r \int \frac{d^3 k}{(2\pi)^3} h_r(\mathbf{k}) F_i^r(\hat{\mathbf{k}}) \exp(-ik(t' - \hat{\mathbf{k}} \cdot \mathbf{r})) [1 - \exp(ikL_i(1 - \hat{\mathbf{k}} \cdot \mathbf{u}_i))]$$

where we have put the earth at the origin, and F_i^r is the detector response, corroborated as,

$$F_i^r(\hat{\mathbf{k}}) = \frac{1}{2} \frac{\hat{\mathbf{u}}^i \hat{\mathbf{u}}^j}{1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{u}}}$$

We use the Stochastic power spectrum

$$\langle h_r(\mathbf{k}, \tau) h_p^*(\mathbf{q}, \tau) \rangle = \frac{8\pi^5}{k^3} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{rp} h_c^2(k, \tau)$$

and noting that the earth term dominates, for two different pulsars ($a \neq b$),

$$[1 - \exp(ikL_a(1 - \hat{\mathbf{k}} \cdot \mathbf{u}_a))] [1 - \exp(-ikL_b(1 - \hat{\mathbf{k}} \cdot \mathbf{u}_b))] \sim 1$$

we simplify as

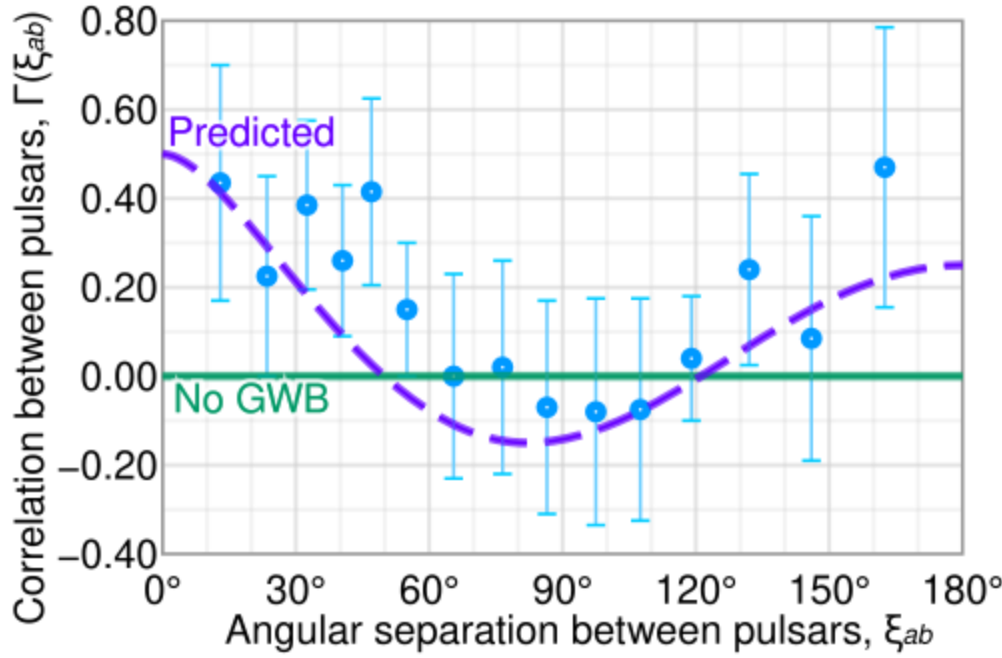
$$\langle R_a(T) R_b(T) \rangle = \mathcal{C}(\theta_{ab}) \int_0^\infty df \frac{h_c^2(f)}{(2\pi)^2 f^3} [1 + \cos(2\pi f(T - t_{\text{ref}}))]$$

where $\mathcal{C}(\theta_{ab})$ solely depends on the angle between the two pulsars, characteristic of the *Hellings & Downs* curve, defined as

$$\mathcal{C}(\theta_{ab}) = \int \frac{d\hat{\mathbf{k}}}{4\pi} \langle F_a^r(\hat{\mathbf{k}}) F_b^r(\hat{\mathbf{k}}) \rangle$$

Similar power spectrum obtained from single source of supermassive black hole with multiple

pulsars.



Stochastic GW Background

Modelling

$$h_c(f) \sim A \left(\frac{f}{f_{\text{ref}}} \right)^{-\alpha}, \quad \alpha = -\frac{2}{3}$$

in terms of the power spectrum of the GW energy density, becomes,

$$\Omega_{\text{GW}} \sim \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f) = \Omega_{\text{GW}}(f_{\text{ref}}) \left(\frac{f}{f_{\text{ref}}} \right)^{\frac{2}{3}}$$

We compute the relative energy density to constrain $\dot{h}$, such that we have

$$\dot{h}(t) = \frac{4\pi^{\frac{2}{3}}}{a(t)r} (GM)^{\frac{5}{3}} \left(\frac{1 + \cos^2 \theta}{2} \right) \frac{d}{dt} [f^{\frac{2}{3}}(t) \cos(2\Phi(t))]$$

such that, we have the integral for

$$\frac{\varrho_{\text{GW}}}{\varrho} = \frac{\pi^{\frac{2}{3}}}{3G\varrho_c} \int \frac{df}{f} f^{\frac{2}{3}} \Omega_{\text{GW}}(f)$$

where the astrophysical modelling of the spectrum conditions the observation.

We have

$$\frac{d\varrho_{\text{GW}}}{d \ln f} \propto \int \frac{dz}{1+z} n(z) \int d^N \lambda \frac{dE}{d \ln f_{\text{GW}}^s}(\Lambda, f)$$

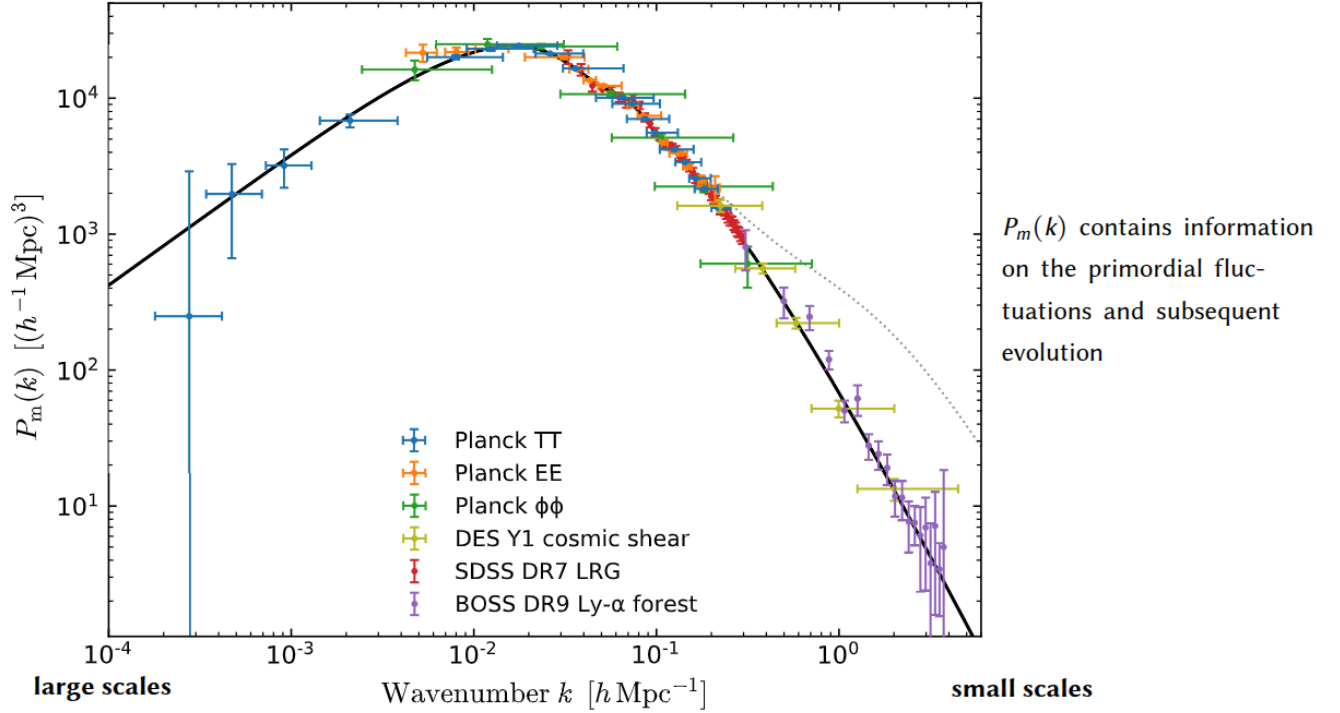
where $f_{\text{GW}} = (1 + z)f$ and $\Lambda = \{\lambda\}$ are the intrinsic parameters. We have the spectral index $\frac{d\rho_{\text{GW}}}{d\ln f} \propto f^{\frac{2}{3}}$.

L01.01 Probing Large Scale Structure

Inhomogeneous Universe

Statistical homogeneous universe at large-scales, has a lot of structure, which evolve with time. CMB anisotropies, and distribution of galaxies.

Matter power spectrum probed from different sources.



Projected correlation functions show clustering.

FLRW Background

Homogeneous and Isotropic metric for $\kappa \rightarrow 0$, given by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -c^2 dt^2 + a^2(t) d\Sigma^2$$

We normalize at current epoch, $a(t_0) = 1$, and defining, redshift observable, $z = \frac{1}{a} - 1$. Substituting in the Einstein equation, we have the Friedmann equations,

$$\left(\frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G}{3} \sum_i P_i$$
$$\frac{\ddot{a}(t)}{a(t)} = -4\pi G \sum_i \left(\rho_i + \frac{3P_i}{c^2} \right)$$

where we assume the energy density and momentum tensor for a perfect fluid.

Combining the equations (conservation of energy density), with $H(t) = \frac{\dot{a}(t)}{a(t)}$, we have

$$\dot{\rho}_i + 3H \left(\rho_i^2 + \frac{P_i}{c^2} \right) = 0$$

such that, we have

$$\rho_i(t) = \rho_{i,0} a^{-3(1+w_i)}$$

where we have the equation state,

$$P_i = w_i \rho_i c^2$$

We often, define the ratios of density to the critical density, $\rho_{\text{crit}} = \frac{3H^2(t)}{8\pi G}$, such that we define

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_{\text{crit}}} = \frac{8\pi G \rho_i(t)}{3H^2(t)}$$

such that, we have the evolution,

$$\Omega_i(t) = \Omega_{i,0} (1+z)^{3(1+w_i)}$$

Thereby, the Hubble equation simplifies, as

$$H^2(t) = H_0^2 [\Omega_{m,0}(1+z)^3 + \Omega_\Lambda + \Omega_{\gamma,0}(1+z)^4]$$

where the universe had initially radiation dominated (Ω_γ), then matter dominated (Ω_m), to currently, Λ dominated (Ω_Λ).

Matter Perturbations

General Perturbations

We have the metric,

$$ds^2 = (\bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}) dx^\alpha dx^\beta$$

where, we have the Einstein equation,

$$\bar{G}_{\alpha\beta} + \delta G_{\alpha\beta} = \frac{8\pi G}{c^4} (\bar{T}_{\alpha\beta} + \delta T_{\alpha\beta})$$

We assume small perturbations, since the equations are *extremely* non-linear.

Thereby, we work with non relativistic **matter** perturbations, with scales smaller than $\frac{c}{H(z)}$, such that we can work in the Newtonian limit for the perturbations. Similar analyses do not work for radiation perturbations.

In physical coordinates $(t, \mathbf{r})$ the fluid and gravity system for non-relativistic matter is

$$\begin{aligned}\dot{\varrho}(t, \mathbf{r}) + \nabla_{\mathbf{r}}(\varrho(t, \mathbf{r})\mathbf{u}(t, \mathbf{r})) &= 0 \\ \dot{\mathbf{u}}(t, \mathbf{r}) + (\mathbf{u}(t, \mathbf{r}) \cdot \nabla_{\mathbf{r}})\mathbf{u}(t, \mathbf{r}) &= -\nabla_{\mathbf{r}}\Phi(t, \mathbf{r}) - \frac{\nabla_{\mathbf{r}}P(t, \mathbf{r})}{\varrho(t, \mathbf{r})} \\ \nabla_{\mathbf{r}}^2\Phi(t, \mathbf{r}) &= 4\pi G \left[\varrho(t, \mathbf{r}) + \bar{\varrho}_{\text{m}}(t) + \frac{3\bar{P}_{\text{m}}(t)}{c^2} \right]\end{aligned}$$

where $\varrho(t, \mathbf{r})$ is the fluid density, and $\mathbf{u}(t, \mathbf{r})$ is the proper velocity of the fluid.

Let us assume dark matter being cold and non relativistic, under the Λ CDM model, using observational and cosmological studies. where, we have the perturbed potential Φ and the pressure P .

Expanding Universe

Firstly, to solve under the framework of the expanding universe, we define the proper/physical coordinates $\mathbf{r}$, in terms of the *comoving* coordinates $\boldsymbol{\chi}$, as

$$\mathbf{r}(t) = a(t)\boldsymbol{\chi}(t)$$

and we define the *density contrast* $\delta(t, \boldsymbol{\chi})$, as

$$\delta(t, \boldsymbol{\chi}) = \frac{\varrho(t, \boldsymbol{\chi})}{\bar{\varrho}_{\text{m}}(t)} - 1$$

where we have averaged over the entire volume. Similarly, we can define the *peculiar velocity*, subtracting the *cosmological expansion velocity*, as

$$\mathbf{v}(t, \boldsymbol{\chi}) = a(t)\frac{d\boldsymbol{\chi}}{dt} = \mathbf{u}(t, \boldsymbol{\chi}) - \frac{\dot{a}(t)}{a(t)}\mathbf{r}$$

and similarly, define the perturbed pressure $p(t, \boldsymbol{\chi}) = P - \bar{P}$ and potential $\phi(t, \boldsymbol{\chi}) = \Phi - \Phi_{\text{FRW}}$. This subtraction of effective background metric separates non-clustering components, where dark energy is assumed to not cluster.

ⓘ Perturbative Fluid Equations

We have the transformation $\mathbf{r}(t) = a(t)\boldsymbol{\chi}(t)$, with $\mathbf{u}(t, \mathbf{r}) = \dot{a}(t)\boldsymbol{\chi} + \mathbf{v}(t, \mathbf{r})$ operators

$$\partial_t|_{\mathbf{r}} = \partial_t|_{\boldsymbol{\chi}} + \partial_t|_{\mathbf{r}} \left(\frac{\mathbf{r}}{a(t)} \right) \cdot \nabla_{\boldsymbol{\chi}} = \partial_t|_{\boldsymbol{\chi}} - \frac{\dot{a}(t)}{a(t)}\boldsymbol{\chi} \cdot \nabla_{\boldsymbol{\chi}}, \text{ and } \nabla_{\mathbf{r}} = \frac{1}{a(t)}\nabla_{\boldsymbol{\chi}}.$$

A. Continuity equation, transforms, as $\dot{\varrho}(t, \mathbf{r}) + \nabla_{\mathbf{r}}(\varrho(t, \mathbf{r})\mathbf{u}(t, \mathbf{r})) = 0$ being rewritten as, $\dot{\varrho}(t, \boldsymbol{\chi}) - \frac{\dot{a}(t)}{a(t)}\boldsymbol{\chi} \cdot \nabla_{\boldsymbol{\chi}}\varrho(t, \boldsymbol{\chi}) + \frac{1}{a(t)}\nabla_{\boldsymbol{\chi}}(\varrho(t, \mathbf{r})[\dot{a}(t)\boldsymbol{\chi} + \mathbf{v}(t, \boldsymbol{\chi})]) = 0$, such that, the expression

simplifies, thereby,

$$\dot{\varrho}(t, \chi) + 3 \frac{\dot{a}(t)}{a(t)} \varrho(t, \chi) + \frac{1}{a(t)} \nabla_{\chi}(\mathbf{v}(t, \chi) \varrho(t, \chi)) = 0$$

An interesting insight is for the static background field $\bar{\varrho}_m(t)$, we have the continuity equation,

$$\dot{\bar{\varrho}}_m(t) + 3 \frac{\dot{a}(t)}{a(t)} \bar{\varrho}_m(t) = 0$$

Now, for perturbations, $\varrho(t, \chi) = \bar{\varrho}_m(t)(1 + \delta(t, \chi))$, such that, when we substitute, thereby,

$$\dot{\delta}(t, \chi) + \frac{1}{a(t)} \nabla_{\chi} [(1 + \delta(t, \chi)) \mathbf{v}(t, \chi)] + \underbrace{\dot{\bar{\varrho}}_m(t) + 3 \frac{\dot{a}(t)}{a(t)} \bar{\varrho}_m(t)}_{=0} = 0$$

B. Euler equation, $\dot{\mathbf{u}}(t, \mathbf{r}) + (\mathbf{u}(t, \mathbf{r}) \cdot \nabla_{\mathbf{r}}) \mathbf{u}(t, \mathbf{r}) = -\nabla_{\mathbf{r}} \Phi(t, \mathbf{r}) - \frac{\nabla_{\mathbf{r}} P(t, \mathbf{r})}{\varrho(t, \mathbf{r})}$. The transformation of $(\mathbf{u}(t, \mathbf{r}) \cdot \nabla_{\mathbf{r}}) \mathbf{u}(t, \mathbf{r})$ as

$\frac{1}{a(t)} ((\dot{a}(t) \chi + \mathbf{v}(t, \chi)) \cdot \nabla_{\chi})(\dot{a}(t) \chi + \mathbf{v}(t, \chi)) = \frac{1}{a(t)} [\dot{a}^2(t) \chi + \dot{a}(t)(\chi \cdot \nabla_{\chi}) \mathbf{v}(t, \chi) + \dot{a}(t) \mathbf{v} + (\mathbf{v} \cdot \nabla_{\chi}) \mathbf{v}]$, and thus combining, the left hand side is,
 $\left(\ddot{a}(t) \chi + \dot{\mathbf{v}}(t, \chi) - \frac{\dot{a}^2(t)}{a(t)} \chi - \frac{\dot{a}(t)}{a(t)} (\chi \cdot \nabla_{\chi}) \mathbf{v}(t, \chi) \right) + \frac{1}{a(t)} [\dot{a}^2(t) \chi + \dot{a}(t)(\chi \cdot \nabla_{\chi}) \mathbf{v}(t, \chi) + \dot{a}(t) \mathbf{v} + (\mathbf{v} \cdot \nabla_{\chi}) \mathbf{v}]$. This simplifies, as $\ddot{a}(t) \chi + \dot{\mathbf{v}}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}(t, \chi) + \frac{1}{a(t)} (\mathbf{v}(t, \chi) \cdot \nabla_{\chi}) \mathbf{v}(t, \chi)$. The right hand side transforms as $-\frac{1}{a(t)} \nabla_{\chi} \Phi(t, \chi) - \frac{1}{a(t) \varrho} \nabla_{\chi} P(t, \chi)$. Thereby, the entire equation, is

$$\dot{\mathbf{v}}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}(t, \chi) + \frac{1}{a(t)} (\mathbf{v}(t, \chi) \cdot \nabla_{\chi}) \mathbf{v}(t, \chi) + \ddot{a}(t) \chi = -\frac{1}{a(t)} \nabla_{\chi} \Phi(t, \chi) - \frac{\nabla_{\chi} P(t, \chi)}{a(t) \varrho(t, \chi)}$$

Now, for perturbations, we use the Friedmann equation, we note $\nabla_{\chi} P(t, \chi) = \nabla_{\chi} p(t, \chi)$, and $\nabla_{\chi} \Phi(t, \chi) = \nabla_{\chi} \phi(t, \chi) + \nabla_{\chi} \Phi_{\text{FRW}}(t, \chi)$, where $\nabla_{\chi} \Phi_{\text{FRW}}(t, \chi) = \ddot{a}(t) \chi$, and hence, we have

$$\dot{\mathbf{v}}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}(t, \chi) + \frac{1}{a(t)} (\mathbf{v}(t, \chi) \cdot \nabla_{\chi}) \mathbf{v}(t, \chi) = -\frac{1}{a(t)} \nabla_{\chi} \phi(t, \chi) - \frac{\nabla_{\chi} p(t, \chi)}{a(t) \bar{\varrho}(1 + \delta(t, \chi))}$$

C. Poisson equation, $\nabla_{\mathbf{r}}^2 \Phi(t, \mathbf{r}) = 4\pi G \left[\varrho(t, \mathbf{r}) + \bar{\varrho}_m(t) + \frac{3\bar{P}_m(t)}{c^2} \right]$, we have the simple transformation,

$$\nabla_{\chi}^2 \Phi(t, \chi) = 4\pi G a^2(t) \left[\varrho(t, \chi) + \bar{\varrho}_m(t) + \frac{3\bar{P}_m(t)}{c^2} \right]$$

and the perturbation, is heavily simplified, as $\nabla_{\chi}^2 \Phi(t, \chi) = \nabla_{\chi}^2 \phi(t, \chi) + \nabla_{\chi}^2 \Phi_{\text{FRW}}(t, \chi)$, where $\nabla_{\chi}^2 \Phi_{\text{FRW}}(t, \chi) = 4\pi G a^2(t) \left[\bar{\varrho} + \bar{\varrho}_m(t) + \frac{3\bar{P}_m(t)}{c^2} \right]$, thereby,

$$\nabla_{\chi}^2 \phi(t, \chi) = 4\pi G \bar{\varrho} a^2(t) \delta(t, \chi)$$

Substituting the perturbations, in the fluid equations, we have

$$\begin{aligned}\dot{\delta}(t, \chi) + \frac{1}{a(t)} \nabla_{\chi} [(1 + \delta(t, \chi)) \mathbf{v}(t, \chi)] &= 0 \\ \dot{\mathbf{v}}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}(t, \chi) + \frac{1}{a(t)} (\mathbf{v}(t, \chi) \cdot \nabla_{\chi}) \mathbf{v}(t, \chi) &= -\frac{1}{a(t)} \nabla_{\chi} \phi(t, \chi) - \frac{\nabla_{\chi} p(t, \chi)}{a(t) \bar{\rho}(1 + \delta(t, \chi))} \\ \nabla_{\chi}^2 \phi(t, \chi) &= 4\pi G \bar{\rho} a^2(t) \delta(t, \chi)\end{aligned}$$

We can separate into relevant quantities, for dark matter and baryons, with the entire set of coupled equations, as

$$\begin{aligned}\dot{\delta}_{\Lambda}(t, \chi) + \frac{1}{a(t)} \nabla_{\chi} [(1 + \delta_{\Lambda}(t, \chi)) \mathbf{v}_{\Lambda}(t, \chi)] &= 0 \\ \dot{\mathbf{v}}_{\Lambda}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}_{\Lambda}(t, \chi) + \frac{1}{a(t)} (\mathbf{v}_{\Lambda}(t, \chi) \cdot \nabla_{\chi}) \mathbf{v}_{\Lambda}(t, \chi) &= -\frac{1}{a(t)} \nabla_{\chi} \phi(t, \chi) \\ \nabla_{\chi}^2 \phi(t, \chi) = 4\pi G a^2(t) [\bar{\rho}_{\Lambda} \delta_{\Lambda}(t, \chi) + \bar{\rho}_{\text{m}} \delta_{\text{m}}(t, \chi)] &= \frac{3H_0^2}{2a(t)} [\bar{\Omega}_{\Lambda,0} \delta_{\Lambda}(t, \chi) + \bar{\Omega}_{\text{m},0} \delta_{\text{m}}(t, \chi)] \\ \dot{\delta}_{\text{m}}(t, \chi) + \frac{1}{a(t)} \nabla_{\chi} [(1 + \delta_{\text{m}}(t, \chi)) \mathbf{v}_{\text{m}}(t, \chi)] &= 0 \\ \dot{\mathbf{v}}_{\text{m}}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}_{\text{m}}(t, \chi) + \frac{1}{a(t)} (\mathbf{v}_{\text{m}}(t, \chi) \cdot \nabla_{\chi}) \mathbf{v}_{\text{m}}(t, \chi) &= -\frac{1}{a(t)} \nabla_{\chi} \phi(t, \chi) - \frac{\nabla_{\chi} p_{\text{m}}(t, \chi)}{a(t) \bar{\rho}_{\text{m}}(1 + \delta_{\text{m}}(t, \chi))}\end{aligned}$$

We note that the perturbed gravitational potential $\phi(t, \mathbf{x})$ arises from dark matter and baryonic matter, and we have the respective densities.

We have the approximation, $\Omega_{\Lambda} \sim 6\Omega_{\text{m}}$, and $\delta_{\Lambda} \gtrsim \delta_{\text{m}}$, such that we have the gravitational potential sourced by dark matter, and baryons moving in the field of dark matter.

Linear Perturbations

We have, in first order of perturbations, for dark matter, which are decoupled completely from baryonic matter, as

$$\begin{aligned}\dot{\delta}_{\Lambda}(t, \chi) + \frac{1}{a(t)} \nabla_{\chi} \mathbf{v}_{\Lambda}(t, \chi) &= 0 \\ \dot{\mathbf{v}}_{\Lambda}(t, \chi) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}_{\Lambda}(t, \chi) &= -\frac{1}{a(t)} \nabla_{\chi} \phi(t, \chi) \\ \nabla_{\chi}^2 \phi(t, \chi) &= \frac{3H_0^2}{2a(t)} \bar{\Omega}_{\Lambda,0} \delta_{\Lambda}(t, \chi)\end{aligned}$$

to independently, have the equation for dark matter, as

$$\ddot{\delta}_{\Lambda}(t, \chi) + 2\frac{\dot{a}(t)}{a(t)} \dot{\delta}_{\Lambda} = \frac{3H_0^2}{2a^3(t)} \bar{\Omega}_{\Lambda,0} \delta_{\Lambda}(t, \chi)$$

For a flat universe, we have $a(t) \rightarrow 1$, such that, $\ddot{\delta}_\Lambda(t, \chi) = \frac{3H_0^2}{2} \bar{\Omega}_{\Lambda,0} \delta_\Lambda(t, \chi)$, leading to exponential solutions in the density contrast, such that $\delta_\Lambda(t, \chi) \sim A \exp(t) + B \exp(-t)$ such that the exponential decay dies at large time scales, such that we have an effective **gravitational instability**, such that small perturbations are exponentially blown up.

We have variable separation, such that

$$\delta_\Lambda(t, \chi) = D(t) f(\chi)$$

such that we have the evolution equation,

$$\ddot{D}(t) + 2 \frac{\dot{a}(t)}{a(t)} D = \frac{3H_0^2}{2a^3(t)} \bar{\Omega}_{\Lambda,0} D(t)$$

which has solution $D(t) \propto H(a(t))$, which is not interesting, in an expanding universe. The other solution, can be found by the method of *Wronskian*, such that, in the limit $\Lambda \rightarrow 0$, such that we have $D(t) \propto a(t)$. We assume $\Lambda \rightarrow 0$ further, such that, we have the generic solution, $\delta_\Lambda(t, \chi) = D_+(a(t)) f(\chi)$, such that we normalize, $D_+(a(t_0)) = 1$.

The peculiar velocity, can be found out from the continuity equation, as

$$v_\Lambda(t, \chi) \propto a(t) \dot{D}_+(a(t)) \propto a(t) D_+(a(t)) H(a(t)) \mathcal{F}(a(t))$$

such that, we have

$$\mathcal{F}(a(t)) = \frac{d \ln D_+(a(t))}{d \ln a(t)}$$

such that, we have the relation $\mathcal{F}(z) \approx \Omega_{m,0}^{\frac{t}{z}}(z)$.

Further, we can substitute the perturbed potential sourced primarily by dark matter, into the baryonic matter field, as,

$$\ddot{\delta}_m(t, \chi) + 2 \frac{\dot{a}(t)}{a(t)} \dot{\delta}_m(t, \chi) - \left(\frac{a(t)}{a(t)} \right)^2 \nabla_\chi^2 \delta_m(t, \chi) = \frac{3H_0^2}{2a^3(t)} \Omega_{\Lambda,0} \delta_m(t, \chi)$$

and we also require an equation of state, for complete set of equations. We decompose into Fourier modes, as

$$\begin{aligned} \delta_m(t, \mathbf{k}) &= \int d^3 \chi \delta_m(t, \chi) \exp(-i \mathbf{k} \cdot \chi) \\ \delta_m(t, \chi) &= \frac{1}{(2\pi)^3} \int d^3 k \delta_m(t, \mathbf{k}) \exp(i \mathbf{k} \cdot \chi) \end{aligned}$$

such that, we have,

$$\ddot{\delta}_m(t, \mathbf{k}) + 2 \frac{\dot{a}(t)}{a(t)} \dot{\delta}_m(t, \mathbf{k}) + \left(\frac{a(t)}{a(t)} \right)^2 k^2 \delta_m(t, \mathbf{k}) = \frac{3H_0^2}{2a^3(t)} \Omega_{\Lambda,0} \delta_m(t, \mathbf{k})$$

where we assume for small $\mathbf{k}$, the linearized equation for baryonic density contrast $\delta_m(t, \mathbf{k})$, sourced by both its own gravity and that of dark matter, is

$$\ddot{\delta}_m(t, \mathbf{k}) + 2\frac{\dot{a}(t)}{a(t)}\dot{\delta}_m(t, \mathbf{k}) - \frac{c_s^2}{a^2(t)}\nabla_\chi^2\delta_m(t, \mathbf{k}) = 4\pi G(\bar{\rho}_\Lambda(t)\delta_\Lambda(t, \mathbf{k}) + \bar{\rho}_m(t)\delta_m(t, \mathbf{k}))$$

Here, $c_s^2 = \frac{\partial p_m}{\partial \rho_m}$ is the square of the sound speed of the baryonic fluid. Transforming to Fourier space by replacing ∇_χ^2 with $-k^2$, where k is the comoving wavenumber, we get the equation for a single mode $\delta_m(t, \mathbf{k})$, as

$$\ddot{\delta}_m(t, \mathbf{k}) + 2\frac{\dot{a}(t)}{a(t)}\dot{\delta}_m(t, \mathbf{k}) + \frac{c_s^2 k^2}{a^2(t)}\delta_m(t, \mathbf{k}) = 4\pi G(\bar{\rho}_\Lambda(t)\delta_\Lambda(t, \mathbf{k}) + \bar{\rho}_m(t)\delta_m(t, \mathbf{k}))$$

This equation beautifully illustrates the cosmic battle between gravity and pressure. The *Jeans scale* is the critical scale that separates these two regimes. For a purely baryonic fluid, the *Jeans wavenumber*, k_J , is defined as the scale where baryonic self-gravity is balanced by pressure given as

$$\underbrace{4\pi G\bar{\rho}_m(t)}_{\text{Self-gravity}} \sim \underbrace{\frac{c_s^2 k_J^2}{a^2(t)}}_{\text{Pressure}}$$

The corresponding comoving Jeans length is $\chi_J \sim 1/k_J$. Using $\bar{\rho}_m(t) = \Omega_{m,0} \frac{3H_0^2}{8\pi G} a^{-3}$, we can write

$$k_J^2 = \frac{4\pi G \left(\Omega_{m,0} \frac{3H_0^2}{8\pi G} a^{-3} \right) a^2}{c_s^2} = \frac{3H_0^2 \Omega_{m,0}}{2ac_s^2}$$

Perturbations with $k < k_J$ (scales larger than the Jeans length) are dominated by gravity and will grow, while those with $k > k_J$ (smaller scales) are dominated by pressure, causing them to oscillate as sound waves (Baryon Acoustic Oscillations) instead of collapsing.

We assume that the baryonic perturbations evolve with the dark matter perturbations, which grow as $\delta_\Lambda(t, \mathbf{k}) = D_+(t)\delta_{\Lambda,i}(\mathbf{k})$. This implies the time-derivative part of the baryonic equation can be related to the overall growth of structure

$$\ddot{\delta}_m(t, \mathbf{k}) + 2\frac{\dot{a}(t)}{a(t)}\dot{\delta}_m(t, \mathbf{k}) \approx 4\pi G(\bar{\rho}_\Lambda(t) + \bar{\rho}_m(t))\delta_m(t, \mathbf{k})$$

Substituting this into the full baryonic equation, we have

$$4\pi G(\bar{\rho}_\Lambda(t) + \bar{\rho}_m(t))\delta_m(t, \mathbf{k}) + \frac{c_s^2 k^2}{a^2(t)}\delta_m(t, \mathbf{k}) \approx 4\pi G(\bar{\rho}_\Lambda(t)\delta_\Lambda(t, \mathbf{k}) + \bar{\rho}_m(t)\delta_m(t, \mathbf{k}))$$

which can be simplified as,

$$\left(4\pi G\bar{\varrho}_\Lambda(t) + \frac{c_s^2 k^2}{a^2}(t)\right)\delta_{\text{m}} \approx 4\pi G\bar{\varrho}_\Lambda(t)\delta_\Lambda(t, \mathbf{k})$$

Solving for $\delta_{\text{m}}(t, \mathbf{k})$, gives the relation

$$\delta_{\text{m}}(t, \mathbf{k}) \approx \frac{4\pi G\bar{\varrho}_\Lambda(t)}{4\pi G\bar{\varrho}_\Lambda(t) + \frac{c_s^2 k^2}{a^2(t)}}\delta_\Lambda(t, \mathbf{k}) = \frac{\delta_\Lambda(t, \mathbf{k})}{1 + \frac{c_s^2 k^2}{4\pi G\bar{\varrho}_\Lambda a^2}}$$

This expression shows how baryonic perturbations are suppressed relative to dark matter perturbations on small scales. The effective scale for this suppression depends on the dark matter density $\bar{\varrho}_\Lambda$. Thereby, we have

$$\delta_{\text{m}}(t, \mathbf{k}) = \frac{\delta_\Lambda(t, \mathbf{k})}{1 + \chi_{\mathcal{J}}^2 k^2}$$

where $\chi_{\mathcal{J}}^2 = 1/k_{\mathcal{J}}^2$ is defined using the baryonic density $\bar{\varrho}_{\text{m}}(t)$.

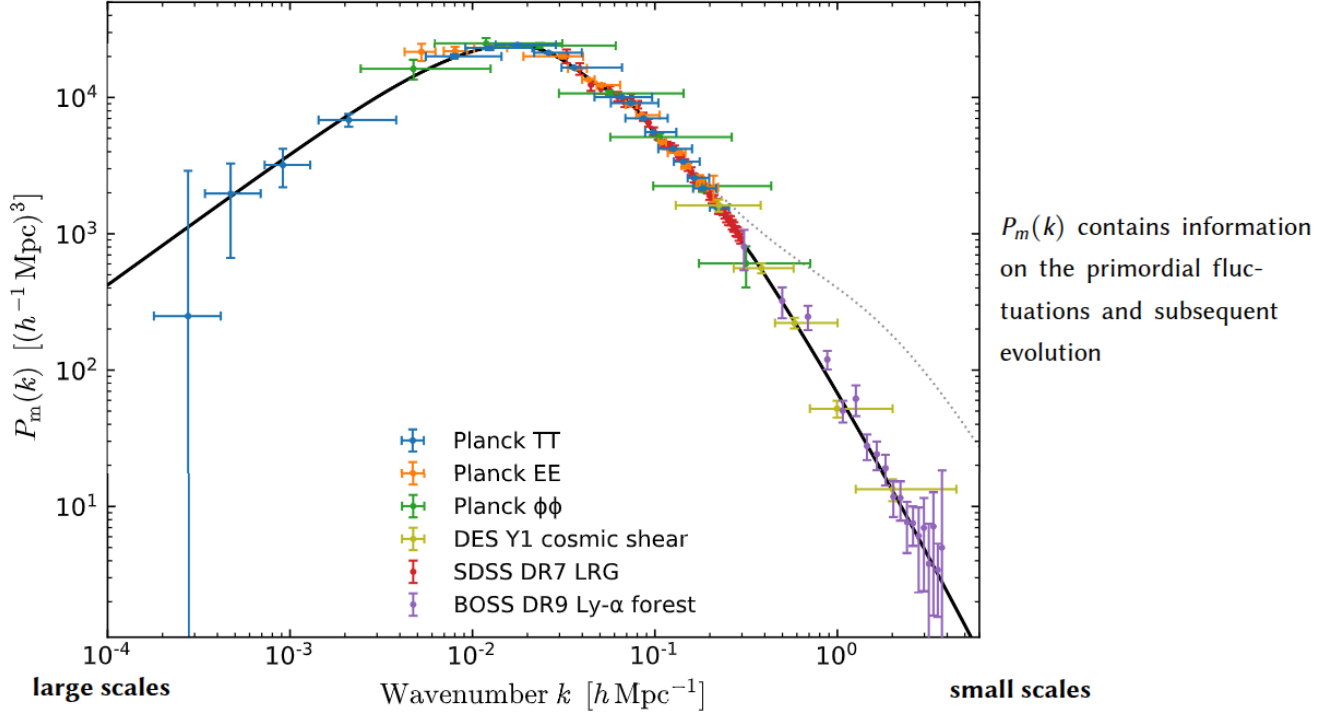
L01.02 Probing Large Scale Structure

Power spectrum

From our separated variables, $\delta \approx \delta_\Lambda(t, \chi) = D(t)f(\chi)$, we have the power spectrum

$$P(\mathbf{k}) \approx |\delta(t - t_0, \mathbf{k})|^2$$

such that, we have $P_{\text{primordial}} \propto k^{n_s-4}$ where $n_s \sim 1$ as the power law variation.



At low k , there is a deviation from observed behavior, because assumptions of matter dominated epoch fails, and length scales grows greater than $\frac{c}{H_0}$.

In the radiation dominated era, we have

$$\ddot{\delta}(t, \chi) + 2 \frac{\ddot{a}(t)}{a(t)} \dot{\delta}(t, \chi) = \frac{\nabla_\chi^2 \phi}{a^2(t)} \rightarrow \frac{k^2 \phi}{a^2(t)}$$

where we have the radiation dominated era, $\delta(t, \chi) \rightarrow \delta_\gamma(t, \chi)$, such that, we input the radiation pressure and solve, to have $\delta_\gamma(t, \chi) \sim \ln a(t)$, thereby, the gravitational instability does not exist.

Gravitational Instability is an extremely rare case of matter dominated universe, at late times, for smaller spatial separations. Perturbations exist inside Hubble radius only for the matter dominated case. Evolution of Hubble radius $\sim \frac{c}{H_0}$. The turnover exists, when the scale is in matter dominated era, reaches the Hubble radius, such that, we have $k \sim \frac{\Omega_{m,0}}{\Omega_\gamma}$.

We can formally, write the power spectral density

$$P(\mathbf{k}) = \mathcal{A}_s k^{n_s} T^2(\mathbf{k})$$

where the power law behavior is truncated by the transfer function $T(\mathbf{k})$ which captures the change of scales and turnover into matter dominated from radiation dominated, and $\mathcal{A}_s$ is the normalization amplitude.

N -Body Simulations

We use tracers in the spacetime, through a series of

$$\varrho_\xi(t, \mathbf{r}) = \sum_i m \delta_D(\mathbf{r} - \mathbf{r}_i(t))$$

such that in comoving coordinates, we have,

$$\varrho_\xi(t, \boldsymbol{\chi}) = \frac{m}{a^3(t)} \sum_i \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i(t))$$

Thereby, the density contrast, is

$$\delta(t, \boldsymbol{\chi}) = \frac{1}{\bar{m}} \sum_i \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i(t)) - 1$$

where $\bar{m} = \frac{\bar{\varrho}_\xi(t, \boldsymbol{\chi})}{m}$. Further, for the velocity, we have

$$\mathbf{u}(t, \mathbf{r}) = \frac{\sum_i \dot{\mathbf{r}} \delta_D(\mathbf{r} - \mathbf{r}_i(t))}{\sum_i \delta_D(\mathbf{r} - \mathbf{r}_i(t))}$$

where we have $\dot{\mathbf{r}} = \dot{a}(t)\boldsymbol{\chi} + a(t)\dot{\boldsymbol{\chi}}$, such that, we have,

$$\mathbf{u}(t, \boldsymbol{\chi}) = \frac{\dot{a}(t)}{a(t)} \mathbf{r}(\boldsymbol{\chi}) + a(t) \frac{\sum_i \dot{\boldsymbol{\chi}} \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i(t))}{\sum_i \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i(t))}$$

where, we shall define,

$$\mathbf{v}(t, \boldsymbol{\chi}) = a(t) \frac{\sum_i \dot{\boldsymbol{\chi}} \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i(t))}{\sum_i \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i(t))}$$

Substituting the above in Euler's equation, we have the effective Newton's II Law,

$$\ddot{\boldsymbol{\chi}}_i(t) + 2 \frac{\dot{a}(t)}{a(t)} \dot{\boldsymbol{\chi}}_i(t) = - \frac{1}{a^2(t)} \nabla_{\boldsymbol{\chi}} \phi(\boldsymbol{\chi}_i)$$

Rewriting, we have $\mathbf{v}_i(t) = a(t)\dot{\boldsymbol{\chi}}_i(t)$, and the effective equation

$$\dot{\mathbf{v}}_i(t) = - \frac{\dot{a}(t)}{a(t)} \mathbf{v}_i(t) - \frac{1}{a^2(t)} \nabla_{\boldsymbol{\chi}} \phi(\boldsymbol{\chi}_i)$$

where we have to specify the evolution of ϕ through the Poisson's equation,

$$\nabla_{\chi}^2 \phi = \frac{3H_0^2}{2a(t)} \Omega_{\xi,0} \delta_{\xi}(t, \chi)$$

which are completely non-linear, where we have the solutions in terms of the Green's function,

$$\phi(t, \chi) = -\frac{3H_0^2}{8\pi a(t)} \Omega_{\xi,0} \int d^3x' \frac{\delta_{\xi}(t, \chi')}{|\chi - \chi'|} = -\frac{\mathcal{G}}{a(t)} \sum_i \frac{m}{|\chi - \chi'|}$$

Linear Analysis

Define a new potential,

$$\Psi(t, \chi) = \frac{2}{3H_0^2 \Omega_{\xi,0}} \frac{a(t)}{D_+(a(t))} \phi(t, \chi)$$

from the earlier potential $\phi(t, \chi)$, such that, in the linear case, $\Psi(t, \chi)$ is constant, through evolution, such that $\Psi(t, \chi) = \Psi_0(\chi)$.

Substituting for the velocity equation,

$$\dot{\mathbf{v}}_i(t) + \frac{\dot{a}(t)}{a(t)} \mathbf{v}_i(t) = -\frac{3H_0^2 D_+(a(t))}{2a(t)} \Omega_{\xi,0} \nabla_{\chi} \Psi_0(\chi_i)$$

Using our relation, $\mathbf{v}_i(t) = a(t) \dot{D}_+(a(t)) \mathbf{u}_i(t)$. On solving, we have

$$\mathbf{v}_i(t) = -a(t) \dot{D}_+(a(t)) \nabla_{\chi} \Psi_0(\chi_i)$$

such that we have a linear constant evolution, with a proxy time $D_+(a(t))$, we have $\chi_i = \mathbf{q}_i - D_+(a(t)) \nabla_{\chi} \Psi_0(\chi_i)$. On a variable transformation, by first order perturbation, we have $\nabla_{\chi} \Psi_0(\chi_i) \approx \nabla_{\mathbf{q}} \Psi_0(\mathbf{q}_i)$, where the *Lagrange transformation* leads to,

$$\chi(\mathbf{q}) = \mathbf{q} - D_+(a(t)) \nabla_{\mathbf{q}} \Psi_0(\mathbf{q})$$

This is an effective field equation, and motion along the tracer particle, termed as the **Zel'dovich approximation**. Accurate even in some non-linear cases.

Under the transformation, we have conservation,

$$\varrho_{\xi}(t, \chi(t, \mathbf{q})) [a^3(t) d^3\chi] = \varrho_{\xi}(t, \mathbf{q}) [a_0^3(t) d^3\mathbf{q}]$$

such that, we have the relation through the Jacobian $\mathcal{J}(\mathbf{q}, \chi(\mathbf{q})) = \frac{\partial \chi^{\alpha}}{\partial q^{\beta}}$,

$$\varrho_{\xi}(t, \mathbf{q}) = \frac{a_0^3}{a^3(t)} \frac{\varrho_{\xi}(t, \chi(t, \mathbf{q}))}{|\mathcal{J}(\mathbf{q}, \chi(\mathbf{q}))|}$$

Hence, the density functional, in first order, is

$$\varrho_\xi(t, \mathbf{q}) = \frac{a_0^3}{a^3(t)} \frac{\varrho_0(\mathbf{q})}{\left| \delta^{\alpha\beta} - D_+(a(t)) \frac{\partial^2 \Psi_0(\chi)}{\partial q^\alpha \partial q^\beta} \right|}$$

where we have the *deformation tensor*, $\frac{\partial^2 \Psi_0(\chi)}{\partial q^\alpha \partial q^\beta}$. In linear case, we approximate, as

$$\varrho_\xi(t, \mathbf{q}) = \frac{a_0^3}{a^3(t)} \varrho_0(\mathbf{q}) [1 + D_+(a(t)) \nabla_{\mathbf{q}}^2 \Psi_0]$$

We express the eigenvalues of the deformation tensor, $\frac{\partial^2 \Psi_0(\chi)}{\partial q^\alpha \partial q^\beta}$, as λ_1 , λ_2 , and λ_3 , such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, such that, we note

$$\varrho_\xi(t, \mathbf{q}) = \frac{a_0^3}{a^3(t)} \frac{\varrho_0(\mathbf{q})}{[1 - D_+(a(t))\lambda_1(\mathbf{q})][1 - D_+(a(t))\lambda_2(\mathbf{q})][1 - D_+(a(t))\lambda_3(\mathbf{q})]}$$

where we have the behavior dependent on the eigenvalues. In the principle direction of the largest eigenvalue λ_1 , we have *pancake* structures, further plane clustering produces *filaments*, and we have generic *halos*.

Spherical Collapse

For the non linear spherically symmetric collapse, we have

$$\chi_i(t) = \frac{R(t)}{a(t)} \frac{a_0}{R_0} \mathbf{q}_i$$

such that, we have $q_i = \frac{R_0}{a_0}$, and we can substitute in the evolutionary equation

$$\ddot{\chi}_i + 2 \frac{\dot{a}(t)}{a(t)} \dot{\chi}_i = - \frac{\nabla_{\chi} \phi(t, \chi)}{a^2(t)}$$

to arrive at

$$\ddot{R}(t) = - \frac{GM}{R^2} - \frac{4\pi G}{3} \left(\bar{\varrho}_{\max} + \frac{3\bar{P}_{\max}}{c^2} \right) \simeq - \frac{GM}{R^2} + H_0^2 \Omega_\Lambda R(t)$$

which shows an effective repulsive effect due to gravity.

We can assume $\Lambda \approx 0$ in the matter dominated universe, such that we set the boundary conditions, $a(t) = a_0$, and have a shell expanding with Hubble radius, with $\dot{R}_0 \simeq H_0 R_0$, such that,

$$\delta(t) = \frac{1}{\varrho_m(t)} \frac{3M}{4\pi R^3(t)} - 1 = \frac{2GM}{\Omega_{m,0} H_0^2} \frac{a^3(t)}{R^3(t)}$$

For matter dominated, we have $a(t) \propto t^{\frac{2}{3}}$, such that, $\frac{\dot{a}(t)}{a(t)} \sim \frac{2}{3t}$. Energy condition, required is

$$E = -\frac{GM}{R_0} \delta_0 < 0$$

such that we need excess matter density, for matter collapse.

Since we have the energy relation,

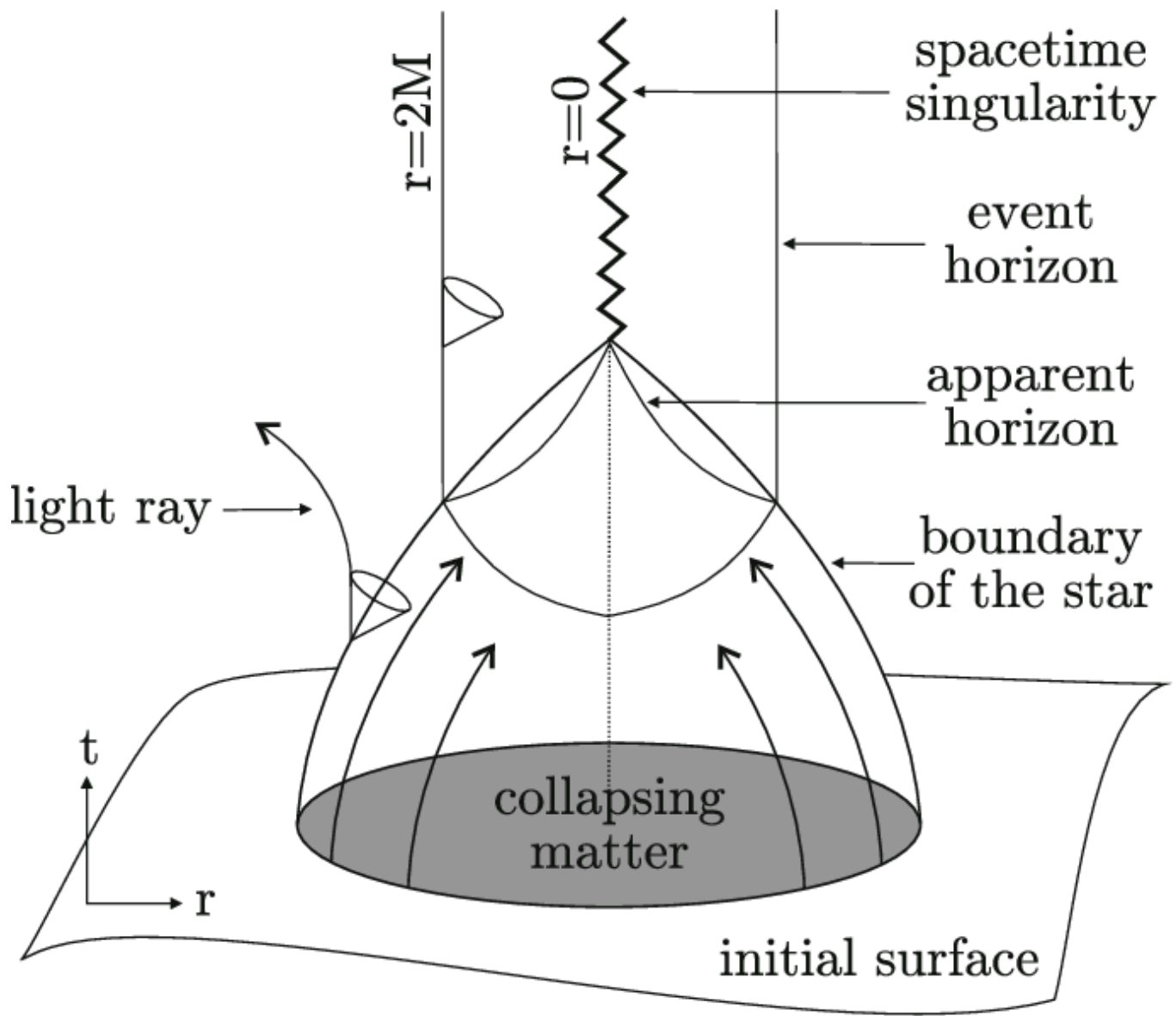
$$E = \frac{1}{2} \dot{R}^2 - \frac{GM}{R}$$

we can explicitly parameterize, $R = A(1 - \cos \theta)$ and $t = B(\theta - \sin \theta)$, where it can be derived that, $A = \frac{GM}{2|E|}$, and $B = \frac{GM}{(2|E|)^{\frac{3}{2}}}$, such that we have the non linear density contrast,

$$\delta_{\text{NL}} = \frac{\rho}{\bar{\rho}} - 1 = \frac{\frac{M}{\frac{4}{3}\pi A^3(1-\cos\theta)^3}}{\frac{1}{6\pi G B^2(\theta-\sin\theta)^2}} - 1 = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3} - 1$$

where we use $A^3 = GMB^2$, which is independent of the energy, thereby, the collapse is possible independent of energy, when there exists a turnover at maximum R .

For a completely linear theory, we expect no collapse, thereby, we expand $t \rightarrow 0$, $\theta \rightarrow 0$, such that, we have the linear density contrast, $\delta_{\text{L}}(t) \simeq \frac{3\theta^2}{20}$, which can be written, as $\delta_{\text{L}}(t) \simeq \frac{3}{20} \left(\frac{6t}{8}\right)^{\frac{2}{3}}$, which is reasonable given matter dominated universe. Collapse exists, in a linear excess of the density contrast.



We can apply the virial theorem at collapse, such that the virial overdensity δ_{vir} is evaluated by noting the virial radius is half the radius at turnover, $R_{\text{vir}} = A$, such that, we have $\delta_{\text{vir}} = 18\pi^2 - 1$.

L02.01 Probing Large Scale Structure

Statistical Description

$\varrho(\boldsymbol{\chi})$ as the realization from the random field ensemble $\varrho_{(a)}(\boldsymbol{\chi})$. We have the ensemble average for a point $\boldsymbol{\chi}_i$, such that

$$\langle \varrho(\boldsymbol{\chi}_i) \rangle = \frac{1}{N_{\text{ensemble}}} \sum_{a=1}^{N_{\text{ensemble}}} \varrho_{(a)}(\boldsymbol{\chi}_i)$$

For a homogeneous universe, we need the ensemble average to be independent of $\boldsymbol{\chi}_i$, such that, we have

$$\langle \varrho(\boldsymbol{\chi}_i) \rangle = \langle \varrho \rangle$$

Similar to the *Ergodic Hypothesis* in statistical mechanics, we now take the volume average,

$${}^V \varrho_{(a)}(\boldsymbol{\chi}) = \frac{1}{V} \int_V d^3x' \varrho_{(a)}(\boldsymbol{\chi} + \boldsymbol{\chi}')$$

Thereby,

$$\langle {}^V \varrho_{(a)}(\boldsymbol{\chi}) \rangle = \left\langle \frac{1}{V} \int_V d^3x' \varrho_{(a)}(\boldsymbol{\chi} + \boldsymbol{\chi}') \right\rangle = \frac{1}{V} \int_V d^3x' \langle \varrho_{(a)}(\boldsymbol{\chi} + \boldsymbol{\chi}') \rangle = \langle \varrho \rangle$$

as an effective unbiased estimator.

For the effective estimator, we need the dispersion,

$$\left\langle [{}^V \varrho_{(a)}(\boldsymbol{\chi}) - \langle \varrho \rangle]^2 \right\rangle_{V \rightarrow \infty} \rightarrow 0$$

which can be shown by noting that, at large scales, for the volume averaged elements, we have an effective drop in correlation. Thus, at some specific scale, we have negligible dispersion.

We have, at large volumes,

$$\lim_{V \rightarrow \infty} {}^V \varrho_{(a)}(\boldsymbol{\chi}) = \bar{\varrho} = \langle \varrho \rangle$$

as the background fluctuation field, spatially independent. Similarly, the density contrast is

$$\delta(\boldsymbol{\chi}) = \frac{\varrho(\boldsymbol{\chi})}{\langle \varrho \rangle} - 1$$

with $\langle \delta(\boldsymbol{\chi}) \rangle = \bar{\delta} = 0$.

Two-Point Correlation Function

We represent the density field by effective tracers,

$$\varrho_{\text{T}}(\boldsymbol{\chi}) = m_{\text{T}} \sum_{i=1}^{N_{\text{T}}} \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i)$$

with the volume average,

$$\langle \varrho_{\text{T}}(\boldsymbol{\chi}) \rangle = \bar{\varrho}_{\text{T}} = m_{\text{T}} \frac{1}{V} \int d^3x' \sum_{i=1}^{N_{\text{T}}} \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_i) = m_{\text{T}} \frac{N_{\text{T}}}{V} = m_{\text{T}} \bar{n}_{\text{T}}$$

For a point $\boldsymbol{\chi}$, the probability of finding a point in δV , is given by,

$$\mathcal{P}_1(\boldsymbol{\chi}) = \lim_{\delta V \rightarrow 0} \frac{\langle \varrho_{\text{T}}(\boldsymbol{\chi}) \delta V \rangle}{m_{\text{T}}} = \lim_{\delta V \rightarrow 0} \frac{\langle \varrho_{\text{T}} \rangle}{m_{\text{T}}} \delta V$$

Further, the probability of finding a point in δV_1 and δV_2 , at points $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2$, is

$$\mathcal{P}_{12}(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \lim_{\delta V_1, \delta V_2 \rightarrow 0} \frac{\langle \varrho_{\text{T}}(\boldsymbol{\chi}_1) \delta V_1 \varrho_{\text{T}}(\boldsymbol{\chi}_2) \delta V_2 \rangle}{m_{\text{T}}^2} = \lim_{\delta V_1, \delta V_2 \rightarrow 0} \frac{\bar{n}_{\text{T}}^2}{\bar{\varrho}_{\text{T}}^2} \langle \varrho_{\text{T}}(\boldsymbol{\chi}_1) \varrho_{\text{T}}(\boldsymbol{\chi}_2) \rangle \delta V$$

such that, we can re-express, as

$$\mathcal{P}_{12}(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \bar{n}_{\text{T}}^2 [1 + \xi_{\text{T}}(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)] \delta V_1 \delta V_2$$

where we have the *two-point correlation function*, given as

$$\xi_{\text{T}}(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \frac{\langle \varrho_{\text{T}}(\boldsymbol{\chi}_1) \varrho_{\text{T}}(\boldsymbol{\chi}_2) \rangle}{\bar{\varrho}_{\text{T}}^2} - 1$$

which is also given in terms of the density contrast, as

$$\xi_{\text{T}}(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \langle \delta_{\text{T}}(\boldsymbol{\chi}_1) \delta_{\text{T}}(\boldsymbol{\chi}_2) \rangle$$

For the tracers, we have

$$\begin{aligned} \langle \varrho_{\text{T}}(\boldsymbol{\chi}_1) \varrho_{\text{T}}(\boldsymbol{\chi}_2) \rangle &= \left\langle m_{\text{T}} \sum_{i=1}^{N_{\text{T}}} \delta_D(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_i) m_{\text{T}} \sum_{j=1}^{N_{\text{T}}} \delta_D(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_j) \right\rangle \\ &= m_{\text{T}}^2 \left\langle \sum_{i=1}^{N_{\text{T}}} \sum_{j=1}^{N_{\text{T}}} \delta_D(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_i) \delta_D(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_j) \right\rangle + m_{\text{T}}^2 \left\langle \sum_{i=1}^{N_{\text{T}}} \sum_{i \neq j}^{N_{\text{T}}} \delta_D(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_i) \delta_D(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_j) \right\rangle \\ &= \bar{\varrho}_{\text{T}}^2 + m_{\text{T}}^2 \langle \delta_D(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2) \rangle \end{aligned}$$

Hence, the two-point correlation is,

$$\xi_T(\chi_1, \chi_2) = \frac{\langle \varrho_T(\chi_1) \varrho_T(\chi_2) \rangle}{\bar{\varrho}_T^2} - 1 = \frac{1}{n_T^2} \delta_D(\chi_1 - \chi_2)$$

where we have neglected the contribution from shot noise, and considered the relative terms.

We now impose homogeneity of the universe to constrain $\xi_T(\chi_1, \chi_2) = \xi_T(\chi_1 - \chi_2)$, as a function of the relative distance. Further, spatial isotropy implies, $\xi_T(\chi_1, \chi_2) = \xi_T(|\chi_1 - \chi_2|)$, removing the angular and directional dependence.

Fourier Space

We have the transformation,

$$\delta_T(\mathbf{k}) = \int d^3x \delta_T(\chi) \exp(-i\mathbf{k} \cdot \chi)$$

with the constraint $\delta_T^*(\mathbf{k}) = \delta_T(-\mathbf{k})$, since $\delta_T(\chi)$ is real. The ensemble average for $\delta_T(\mathbf{k})$, can be seen as

$$\langle \delta_T(\mathbf{k}) \delta_T^*(\mathbf{k}') \rangle = \left\langle \int d^3x \delta_T(\chi) \exp(-i\mathbf{k} \cdot \chi) \cdot \int d^3x' \delta_T(\chi') \exp(i\mathbf{k}' \cdot \chi') \right\rangle$$

which can be simplified, as

$$\begin{aligned} \langle \delta_T(\mathbf{k}) \delta_T^*(\mathbf{k}') \rangle &= \int d^3x d^3x' \underbrace{\langle \delta_T(\chi) \delta_T(\chi') \rangle}_{\xi_T(\chi, \chi')} \exp(-i[\mathbf{k} \cdot \chi - \mathbf{k}' \cdot \chi']) \\ &= \int d^3x d^3x' \xi_T(\chi - \chi') \exp(-i[\mathbf{k} \cdot \chi - \mathbf{k}' \cdot \chi']) \end{aligned}$$

By change of variables, we arrive at,

$$\langle \delta_T(\mathbf{k}) \delta_T^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') \int d^3y \exp(-i\mathbf{k} \cdot \mathbf{y}) \xi_T(\mathbf{y})$$

Define the *Power spectrum*, such that

$$P_T(\mathbf{k}) = \int d^3x \xi_T(\chi) \exp(-i\mathbf{k} \cdot \chi)$$

such that, we have the correlation function Fourier transform. We note,

$$\langle \delta_T(\mathbf{k}) \delta_T^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P_T(\chi)$$

By isotropy, we simplify the power spectral density, as

$$\begin{aligned} P_T(\mathbf{k}) &= \int d^3x \xi_T(|\chi|) \exp(-i\mathbf{k} \cdot \chi) \\ &= \int_0^\infty dx 4\pi x^2 \xi_T(x) \frac{\sin kx}{kx} \end{aligned}$$

thus, the power spectral density $P_T(\mathbf{k}) = P_T(x)$. Similarly, the correlation function,

$$\xi_T(\chi) = \int_0^\infty \frac{dk}{k} \frac{\Delta_T^2(k)}{2\pi^2} \frac{\sin kx}{kx}$$

where $\Delta_T^2(k) = k^3 P_T(k)$ is the *dimensionless power spectrum*, such that it captures the behavior of the effective power in the correlation function, in logarithmic bins of $d \ln k = \frac{dk}{k}$.

For the effective tracers, we have the power spectrum given as, the normalized power spectrum and the contribution arising from shot noise.

Angular Projections

For a comoving distance $\chi(z)$, subtending an effective solid angle Ω , in a small volume δV , we have

$$\chi(z) = c \int_0^z \frac{dz'}{H(z')}$$

such that, the volume element is $\delta V = \chi^2 d\chi d\Omega$.

We have the mass, $\delta M_T = \varrho_T(\chi) \delta V = \varrho_T(\chi, \Omega) \chi^2 d\chi d\Omega$, where we have the projected density on the angular space. Thereby, at an solid angle, we have,

$$\varrho_T|_\Omega = \frac{\delta M_T}{d\Omega} = \int d\chi \chi^2 \varrho_T(\chi, \Omega) = \int dz \frac{\partial \chi}{\partial z} \chi^2 \varrho_T(\chi, \Omega)$$

① Spherical Harmonics

Further, we expand the density contrast in terms of effective spherical harmonics,

$$\delta_T|_\Omega = \frac{\varrho_T|_\Omega}{\bar{\varrho}_T} - 1 = \sum_{\ell=0}^N \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\Omega)$$

The coefficients $a_{\ell m}$ of a function expanded in spherical harmonics are found by projecting the function onto the corresponding spherical harmonic basis function $Y_{\ell m}(\hat{\Omega})$. This is done using the orthogonality property of the spherical harmonics. The expansion is given by

$$\delta_T|_\Omega = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{\Omega})$$

The coefficients are thus calculated as

$$a_{\ell m} = \int d\hat{\Omega} \delta_T|_\Omega Y_{\ell m}^*(\hat{\Omega})$$

where the integral is over the entire solid angle.

We now substitute the definition of the projected density contrast,

$\delta_{\text{T}}|_{\Omega} = \int d\chi \chi^2 S(\chi) \delta_{\text{T}}(\chi, \hat{\Omega})$, into the expression for $a_{\ell m}$,

$$a_{\ell m} = \int d\hat{\Omega} \left[\int_0^{\infty} d\chi \chi^2 S(\chi) \delta_{\text{T}}(\chi, \hat{\Omega}) \right] Y_{\ell m}^*(\hat{\Omega})$$

By swapping the order of integration, we arrive at the expression for $a_{\ell m}$ in terms of the matter density contrast $\delta_{\text{T}}(\chi)$, where the position vector is $\chi = \chi \hat{\Omega}$,

$$a_{\ell m} = \int_0^{\infty} d\chi \chi^2 S(\chi) \int d\hat{\Omega} \delta_{\text{T}}(\chi, \hat{\Omega}) Y_{\ell m}^*(\hat{\Omega})$$

This is the expression for the coefficients in real space.

Let's express the matter density contrast $\delta_{\text{T}}(\chi)$ in terms of its Fourier transform $\tilde{\delta}_{\text{T}}(\mathbf{k})$, as

$$\delta_{\text{T}}(\chi) = \int \frac{d^3 k}{(2\pi)^3} \tilde{\delta}_{\text{T}}(\mathbf{k}) \exp\{i\mathbf{k} \cdot \chi\}$$

We use the Rayleigh plane wave expansion for $\exp\{i\mathbf{k} \cdot \chi\}$ in terms of spherical Bessel functions j_{ℓ} and spherical harmonics $Y_{\ell m}$, given by,

$$\exp\{i\mathbf{k} \cdot \chi\} = 4\pi \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} i^{\ell'} j_{\ell'}(k\chi) Y_{\ell' m'}(\hat{\chi}) Y_{\ell' m'}^*(\hat{k})$$

Substituting this into our expression for $a_{\ell m}$ with $\chi = \chi \hat{\Omega}$ gives

$$a_{\ell m} = \int_0^{\infty} d\chi \chi^2 S(\chi) \int d\hat{\Omega} \left[\int \frac{d^3 k}{(2\pi)^3} \tilde{\delta}_{\text{T}}(\mathbf{k}) \left(4\pi \sum_{\ell' m'} i^{\ell'} j_{\ell'}(k\chi) Y_{\ell' m'}(\hat{\Omega}) Y_{\ell' m'}^*(\hat{k}) \right) \right] Y_{\ell m}^*(\hat{\Omega})$$

Rearranging the integrals and sums, we have

$$a_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \tilde{\delta}_{\text{T}}(\mathbf{k}) \left[4\pi \sum_{\ell' m'} i^{\ell'} Y_{\ell' m'}^*(\hat{k}) \int_0^{\infty} d\chi \chi^2 S(\chi) j_{\ell'}(k\chi) \left(\int d\hat{\Omega} Y_{\ell' m'}(\hat{\Omega}) Y_{\ell m}^*(\hat{\Omega}) \right) \right]$$

The inner integral over the solid angle $d\hat{\Omega}$ is simplified by the orthogonality relation $\int d\hat{\Omega} Y_{\ell' m'}(\hat{\Omega}) Y_{\ell m}^*(\hat{\Omega}) = \delta_{\ell \ell'} \delta_{m m'}$. This collapses the sum over ℓ' and m' to a single term where $\ell' = \ell$ and $m' = m$. Thereby,

$$a_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \tilde{\delta}_{\text{T}}(\mathbf{k}) \left[4\pi i^{\ell} Y_{\ell m}^*(\hat{k}) \int_0^{\infty} d\chi \chi^2 S(\chi) j_{\ell}(k\chi) \right]$$

Defining a radial transfer function $\Delta_{\ell}(k) = \int_0^{\infty} d\chi \chi^2 S(\chi) j_{\ell}(k\chi)$, the final expression for the coefficient is

$$a_{\ell m} = 4\pi i^{\ell} \int \frac{d^3 k}{(2\pi)^3} \tilde{\delta}_{\text{T}}(\mathbf{k}) \Delta_{\ell}(k) Y_{\ell m}^*(\hat{k})$$

The angular power spectrum $\mathcal{C}_\ell$ is defined from the two-point correlation function of the harmonic coefficients

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \mathcal{C}_\ell \delta_{\ell\ell'} \delta_{mm'}$$

where the angle brackets $\langle \cdot \rangle$ denote an ensemble average.

We use the Fourier space expression for $a_{\ell m}$,

$$a_{\ell m} = 4\pi i^\ell \int \frac{d^3 k}{(2\pi)^3} \tilde{\delta}_T(\mathbf{k}) \Delta_\ell(k) Y_{\ell m}^*(\hat{k})$$

and the complex conjugate,

$$a_{\ell' m'}^* = 4\pi (-i)^{\ell'} \int \frac{d^3 k'}{(2\pi)^3} \tilde{\delta}_T^*(\mathbf{k}') \Delta_{\ell'}(k') Y_{\ell' m'}(\hat{k}')$$

The expectation value is then,

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = (4\pi)^2 i^{\ell-\ell'} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \langle \tilde{\delta}_T(\mathbf{k}) \tilde{\delta}_T^*(\mathbf{k}') \rangle \Delta_\ell(k) \Delta_{\ell'}(k') Y_{\ell m}^*(\hat{k}) Y_{\ell' m'}(\hat{k}')$$

The correlation of the Fourier modes of the density contrast defines the matter power spectrum $P(k)$, as defined earlier,

$$\langle \tilde{\delta}_T(\mathbf{k}) \tilde{\delta}_T^*(\mathbf{k}') \rangle = (2\pi)^3 P(k) \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

where $\delta^{(3)}$ is the three-dimensional Dirac delta function. Substituting this into our expression,

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = (4\pi)^2 i^{\ell-\ell'} \int \frac{d^3 k}{(2\pi)^3} P(k) \Delta_\ell(k) \Delta_{\ell'}(k) Y_{\ell m}^*(\hat{k}) Y_{\ell' m'}(\hat{k})$$

The integral over $\mathbf{k}'$ was eliminated by the Dirac delta function, which sets $\mathbf{k}' = \mathbf{k}$.

We now express the integral over $d^3 k$ in spherical coordinates, $d^3 k = k^2 dk d\hat{k}$, thereby,

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \frac{(4\pi)^2}{(2\pi)^3} i^{\ell-\ell'} \int_0^\infty k^2 dk P(k) \Delta_\ell(k) \Delta_{\ell'}(k) \int d\hat{k} Y_{\ell m}^*(\hat{k}) Y_{\ell' m'}(\hat{k})$$

Using the orthogonality of spherical harmonics, $\int d\hat{k} Y_{\ell m}^*(\hat{k}) Y_{\ell' m'}(\hat{k}) = \delta_{\ell\ell'} \delta_{mm'}$, the expression simplifies to

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \frac{16\pi^2}{8\pi^3} i^{\ell-\ell'} \delta_{\ell\ell'} \delta_{mm'} \int_0^\infty k^2 dk P(k) \Delta_\ell(k) \Delta_{\ell'}(k)$$

The Kronecker delta $\delta_{\ell\ell'}$ ensures that the term is non-zero only when $\ell = \ell'$, which makes

the factor $i^{\ell-\ell'} = i^0 = 1$ and $\Delta_{\ell'}(k) = \Delta_{\ell}(k)$. We have,

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \frac{2}{\pi} \delta_{\ell\ell'} \delta_{mm'} \int_0^{\infty} k^2 dk P(k) [\Delta_{\ell}(k)]^2$$

By comparing this result with the definition $\langle a_{\ell m} a_{\ell' m'}^* \rangle = \mathcal{C}_{\ell} \delta_{\ell\ell'} \delta_{mm'}$, we can identify the angular power spectrum $\mathcal{C}_{\ell}$,

$$\mathcal{C}_{\ell} = \frac{2}{\pi} \int_0^{\infty} k^2 dk P(k) \left[\int_0^{\infty} d\chi \chi^2 S(\chi) j_{\ell}(k\chi) \right]^2$$

When we have redshift information through a selection function, we have

$$\varrho_{\mathcal{T}}|_{\Omega} = \int d\chi \chi^2 S(\chi) \varrho_{\mathcal{T}}(\chi, \Omega)$$

with appropriate normalization for the selection function

$$\int d\chi \chi^2 S(\chi) = 1$$

For the projected correlation function, we have

$$\omega_{\mathcal{T}}|_{\Omega_1, \Omega_2} = \frac{\langle \varrho_{\mathcal{T}}|_{\Omega_1} \varrho_{\mathcal{T}}|_{\Omega_2} \rangle}{\bar{\varrho}_{\mathcal{T}}^2} - 1$$

where the average of the projected function is equal to the average of the density function. Evaluating it, we have,

$$\omega_{\mathcal{T}}|_{\Omega_1, \Omega_2} = \frac{1}{\bar{\varrho}_{\mathcal{T}}^2} \left\langle \int d\chi_1 \chi_1^2 S(\chi_1) \varrho_{\mathcal{T}}(\chi_1, \Omega_1) \cdot \int d\chi_2 \chi_2^2 S(\chi_2) \varrho_{\mathcal{T}}(\chi_2, \Omega_2) \right\rangle - 1$$

where we can use explicit assumption of *flat space*, by approximating the relative vector, as

$$\chi_{12} = \sqrt{\chi_1^2 + \chi_2^2 - 2\chi_1\chi_2 \cos \theta} \approx \chi_1 - \chi_2$$

such that we can simplify extensively.

Smoothed Cosmic Fields

Smoothed density field,

$$\delta_{\mathcal{T}}|_X(\mathbf{x}) = \int d^3y \delta_{\mathcal{T}}(\mathbf{y}) W_X(\mathbf{y} - \mathbf{x})$$

with an appropriate window function W_X for a manifold X .

We have the convolution with the Window function, being simplified as a product in the Fourier space as

$$\delta_{\text{T}|X}(\mathbf{k}) = \delta_{\text{T}}(\mathbf{k}) \tilde{W}_X(-\mathbf{k})$$

We have the ensemble average $\langle \delta_{\text{T}|X} \rangle = 0$, and $\langle \delta_{\text{T}|X}^2 \rangle$ relating the power spectral density, as

$$\langle \delta_{\text{T}|X}^2 \rangle = \int \frac{d^3 k}{(2\pi)^3} P_{\text{T}}(\mathbf{k}) |\tilde{W}_X(\mathbf{k})|^2$$

If we have a Gaussian uncertainty smoothening, we have

$$\varrho_{\text{T}}(\chi) \sim \sum_i \exp \left[-\frac{1}{2} \frac{|\chi_i - \chi^2|}{2\sigma_{\chi}^2} \right]$$

where we normalize the Gaussian, with the expression for the total number of tracers in the observed volume giving ρ_{T} , such that,

$$\int d^3 \chi \varrho_{\text{T}}(\chi) = \rho_{\text{T}}$$

thereby, giving us,

$$\varrho_{\text{T}}(\chi) = \frac{1}{(2\pi\sigma_{\chi})^{\frac{3}{2}}} \sum_i \exp \left[-\frac{1}{2} \frac{|\chi_i - \chi^2|}{2\sigma_{\chi}^2} \right]$$

The correlation function, and the power spectrum, due to the convolution of the Gaussian windowing, are suppressed by the Gaussian noise, by the exponential factor $\exp(-\sigma_{\chi}^2 k^2)$, where the suppression is large at large k which corresponds to smaller scales.

L03.01 Probing Large Scale Structures

Simulations

Gaussian Field

For N points, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, we have $\delta_{\mathbf{T}}(\mathbf{x}) \rightarrow \delta_{\mathbf{T},i}$ for $i = 1, 2, \dots, n$. We have the probability distribution,

$$\mathcal{P}(\delta_{\mathbf{T},1}, \delta_{\mathbf{T},2}, \dots, \delta_{\mathbf{T},n}) = \frac{1}{(2\pi)^{\frac{N}{2}} |\mathbf{C}|^{\frac{1}{2}}} \exp \left[-\frac{1}{2} \sum_{i,j=1}^N \delta_{\mathbf{T},i} \mathbf{C}_{ij}^{-1} \delta_{\mathbf{T},j} \right]$$

is a linear, implied *Gaussian* random field, with $\mathbf{C}_{ij} = \langle \delta_{\mathbf{T},i} \delta_{\mathbf{T},j} \rangle$ is the covariance matrix, given equivalently by the two-point correlation function, $\mathbf{C}_{ij} = \xi(\mathbf{x}_i, \mathbf{x}_j)$. For a Gaussian random field, the higher order moments are completely determined by the two-point correlation function, and the mean.

We have the the individual probability distributions, for the initial densities, given by,

$$\mathcal{P}(\delta_{\mathbf{T}}) = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{T}}} \exp \left[-\frac{1}{2} \frac{\delta_{\mathbf{T}}^2}{\sigma_{\mathbf{T}}^2} \right]$$

where $\sigma_{\mathbf{T}}^2 = \langle \delta_{\mathbf{T}}^2 \rangle$ as the variance, which shows the power spectrum, as an effective diagonal covariance matrix. We further, perform the smoothing through appropriate window functions, where we realize that, the windowed density function, is a weighted sum of Gaussian random densities, such that, we have

$$\mathcal{P}(\delta_{\mathbf{T},X}) = \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{T}}(X)} \exp \left[-\frac{1}{2} \frac{\delta_{\mathbf{T},X}^2}{\sigma_{\mathbf{T}}^2(X)} \right]$$

Further, in the frequency domain, we realize the independence of different wave modes, such that the power spectrum is diagonal, to have

$$\langle \delta(\mathbf{k}_1) \delta(\mathbf{k}_2) \rangle \propto \delta_D(\mathbf{k}_1 + \mathbf{k}_2)$$

Linearly Extrapolated Field

We can expand the complete non-linear spectrum in terms of the linear Gaussian random density field as,

$$\delta_{\mathbf{T}}(z, \mathbf{k}) = D_+(a(z)) \underbrace{\delta_{\mathbf{T}}(z=0, \mathbf{k})}_{\delta_{\mathbf{T}}(\mathbf{k})}$$

where $\delta_T(\mathbf{k})$ is the Gaussian field, with the encapsulation of the transfer function $T(\mathbf{k})$. We have the power spectral density and the variance, given by the evolution,

$$P_T(z, \mathbf{k}) = D_+^2(z) \underbrace{P_{\text{prim}}(\mathbf{k}) T^2(\mathbf{k})}_{P_T(\mathbf{k})}$$

$$\sigma_T(z, X) = D_+^2(z) \sigma_T(X)$$

with the variance dependent on the window function used on the manifold.

Halos

Consider early times, with minimal perturbations, $z_{\text{ini}} \gg 1$, such that, we have

$$\delta_T(z_{\text{ini}}, \mathbf{x}) \rightarrow P_T(z_{\text{ini}}, \mathbf{x})$$

which we further smoothen with a window function $W_X(\mathbf{x})$, such that, we have

$$\delta_{T,X}(z_{\text{ini}}, \mathbf{x}) = \int d^3y \delta_T(z_{\text{ini}}, \mathbf{x} + \mathbf{y}) W_X(\mathbf{y})$$

which we can evolve to present time, such that,

$$\delta_{T,X}(z, \mathbf{x}) = \frac{D_+(a(z))}{D_+(a(z_{\text{ini}}))} \delta_{T,X}(z_{\text{ini}}, \mathbf{x})$$

where we can prescribe an effective mass for the region of radius X , with a spherical window function, such that

$$\tilde{M} = \frac{4\pi}{3} (a_{\text{ini}} X)^3 \varrho_T(z_{\text{ini}})$$

where $a^3 \varrho_{T,X}(z_{\text{ini}})$ gives the current density field, $\bar{\varrho}_T(z)$. We have the critical density for spherical collapse, as δ_{crit} , thereby, we have densities, with larger overdensity to have spherical collapse, as

$$\delta_{T,X}(z, \mathbf{x}) > \delta_{\text{crit}}$$

such that, we note the ensemble fraction where we observe collapse.

We can find the fraction of mass within collapsed halos of mass $M > \tilde{M}$, at z , given by the Gaussian cumulative distribution function,

$$\mathcal{P}_{\text{collapse}}(z, M > \tilde{M}) = \frac{1}{\sqrt{2\pi}\sigma_T(z, X)} \int_{\delta_{\text{crit}}}^{\infty} d\delta_{T,X}(z) \exp \left[-\frac{1}{2} \frac{\delta_{T,X}^2}{\sigma_T^2(z, X)} \right]$$

where we evaluate for overdense regions where we expect spherical collapse. The probability

can be evaluated by the complementary erfc function, such that,

$$\mathcal{P}_{\text{collapse}}(z, M > \tilde{M}) = \frac{1}{2} \text{erfc} \left[\frac{\delta_{\text{crit}}}{\sqrt{2} \sigma_{\text{T}}(z, X)} \right] = \frac{1}{2} \text{erfc} \left[\frac{\delta_{\text{crit}}(z)}{\sigma_{\text{T}}(X)} \right]$$

where we have defined $\delta_{\text{crit}}(z) = \frac{\delta_{\text{crit}}}{D_+(a(z))}$ as the effective critical density at current redshift. Note that, due to not being normalized, we have

$$\lim_{M \rightarrow 0} \mathcal{P}_{\text{collapse}}(z, M > \tilde{M}) = \lim_{M \rightarrow 0} \frac{1}{2} \text{erfc} \left[\frac{\delta_{\text{crit}}(z)}{\underbrace{\sigma_{\text{T}}(X)}_{\sigma_{\text{T}}(M)}} \right] \rightarrow \frac{1}{2}$$

which can be fixed. This relates to the *Press-Schechter* relation.

We have the *mass function* $\frac{\partial n}{\partial M}$ as the number density over mass of halo states, given as

$$\mathcal{P}_{\text{collapse}}(z, M > \tilde{M}) = \frac{1}{\bar{\varrho}_{\text{T}}(z_{\text{ini}})} \int_M^\infty dM' M' \frac{\partial n}{\partial M'}$$

where we can derive analytically, the expression for the mass function, as

$$\frac{\partial n}{\partial M} = \sqrt{\frac{2}{\pi}} \frac{\bar{\varrho}_{\text{T}}(z_{\text{ini}})}{M} \left| \frac{\partial \sigma_{\text{T}}(M)}{\partial M} \right| \frac{\delta_{\text{crit}}(z)}{\sigma_{\text{T}}^2(M)} \exp \left[-\frac{1}{2} \frac{\delta_{\text{crit}}^2(z)}{\sigma_{\text{T}}^2(M)} \right]$$

where we realize that the derivation required Linear Perturbation theory, Spherical collapse, and dynamics of random Gaussian field.

Criticality

We can rewrite the expression of the mass function,

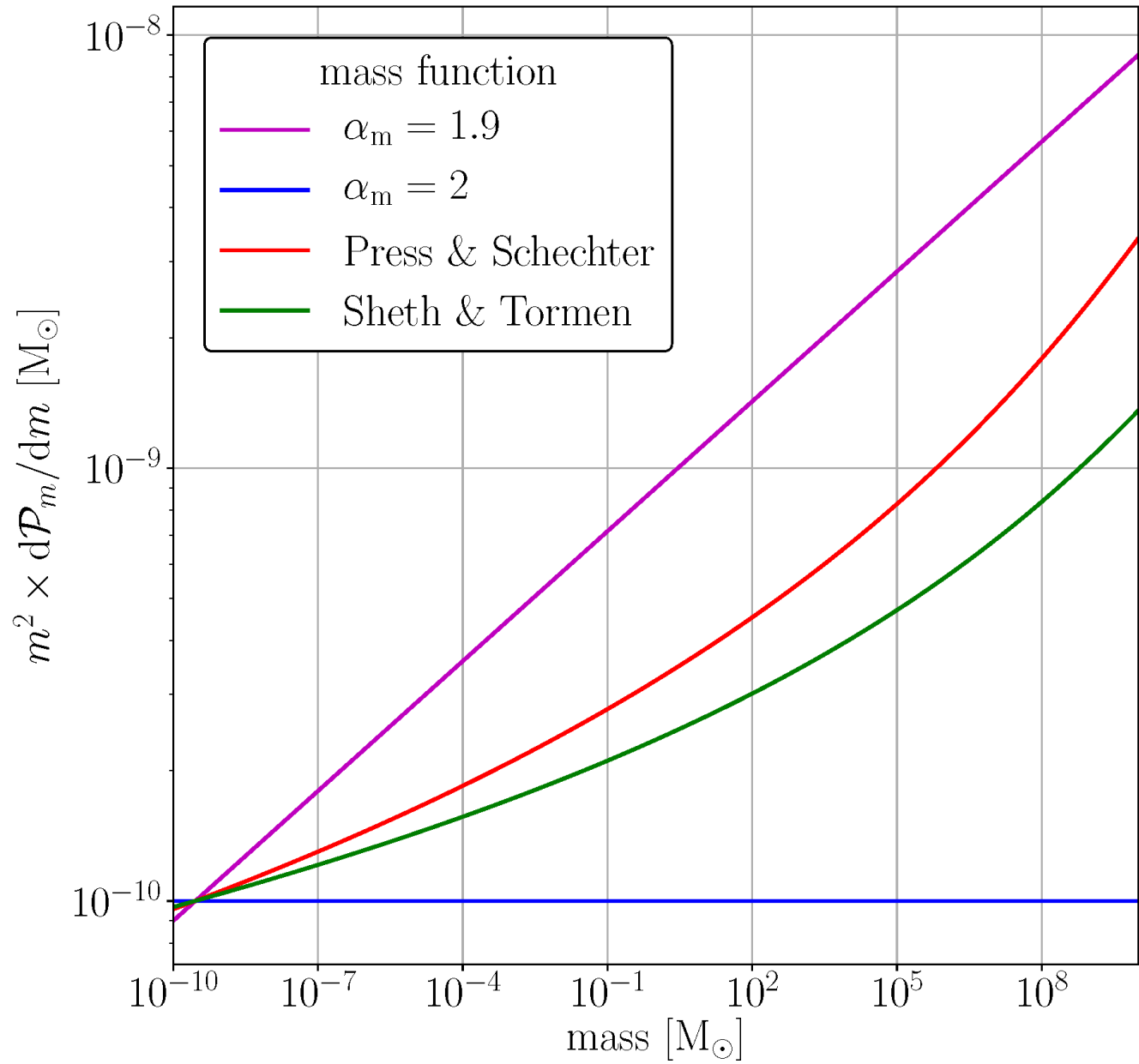
$$\frac{\partial n}{\partial M} = \sqrt{\frac{2}{\pi}} \frac{\bar{\varrho}_{\text{T}}(z_{\text{ini}})}{M} \left| \frac{\partial \ln \sigma_{\text{T}}(M)}{\partial M} \right| \nu \exp \left[-\frac{1}{2} \nu^2 \right]$$

where $\nu = \frac{\delta_{\text{crit}}(z)}{\sigma_{\text{T}}(M)}$.

Let M_* , such that $\sigma_{\text{T}}(M_*) = \delta_{\text{crit}}(z)$, such that, we have for $M < M_*$, we have a power law behavior, and for $M > M_*$, we have an exponential cutoff, that is $\nu < 1$ and $\nu > 1$ respectively.

The mass function integrates to the total number of halos in the universe, where we assume collisionless mass function. Here all properties of the halos are assumed to be dependent solely

on the mass. Similar analysis for ellipsoidal collapse is *Sheth-Tormen* mass function.



L03.02 Probing Large Scale Structure

Statistics and Correlations

Clustering

We recall, the effective mass function with the number of halos per unit comoving volume, concerning per unit mass, $\frac{\partial n}{\partial M}$ determined as the average

$$\bar{n}_M(z) = \frac{\partial n}{\partial M}$$

where we can define the density contrast,

$$\delta_H(\mathbf{x}, M, z) = \frac{n_M(\mathbf{x}, z)}{\bar{n}_M(z)} - 1$$

We can explain clustering for overdense regions by understanding the *peak-background* split, such that we have smaller push for short wavelength modes, superimposed on longer wavelength modes, for regions of positive density contrast. Formally,

$$n_M(\mathbf{x}, z) = \sqrt{\frac{2}{\pi}} \frac{\bar{\varrho}_T(z_{\text{ini}})}{M} \left| \frac{\partial \ln \sigma_T(M)}{\partial M} \right| \nu \exp \left[-\frac{1}{2} \nu^2 \right]$$

where we have

$$\nu = \frac{\delta_{\text{crit}} - \delta_T(\mathbf{x})}{D_+(a(z))\sigma_T(M)}$$

We now expand, for lower orders,

$$\begin{aligned} n_M(\mathbf{x}, z) &= n_M(\mathbf{x}, z) \Big|_{\delta_T(\mathbf{x})=0} + \frac{\partial n_M(\mathbf{x}, z)}{\partial \delta_T(\mathbf{x})} \Big|_{\delta_T(\mathbf{x})=0} \delta_T(\mathbf{x}) + \mathcal{O}(\delta_T^2(\mathbf{x})) \\ &= \bar{n}_M(\mathbf{x}) + \frac{\partial n_M(\mathbf{x}, z)}{\partial \nu} \frac{\partial \nu}{\partial \delta_T(\mathbf{x})} \Big|_{\delta_T(\mathbf{x})=0} \delta_T(\mathbf{x}) + \mathcal{O}(\delta_T^2(\mathbf{x})) \\ &= \bar{n}_M(\mathbf{x}) \left[1 + \frac{\nu^2 - 1}{\delta_{\text{crit}}} \delta_T(\mathbf{x}) \right] + \mathcal{O}(\delta_T^2(\mathbf{x})) \end{aligned}$$

Thereby, the density contrast, for halos at a particular redshift, is given by,

$$\delta_H(\mathbf{x}, M) = \frac{\nu^2 - 1}{\delta_{\text{crit}}} \delta_T(\mathbf{x})$$

where we have the definition, $b(M) = \frac{\nu^2 - 1}{\delta_{\text{crit}}}$, as the linear halo bias. Thereby, the correlation

function,

$$\xi_H(\mathbf{x}_1 - \mathbf{x}_2 | M_1, M_2) = \langle \delta_H(\mathbf{x}_1, M_1) \delta_H(\mathbf{x}_2, M_2) \rangle = b(M_1) b(M_2) \xi_T(\mathbf{x}_1 - \mathbf{x}_2)$$

If we were to look at the correlation density profile for masses $M > M_{\min}$, we have,

$$\xi_H(\mathbf{x}_1 - \mathbf{x}_2 | M_{\min}) = \frac{\int_{M_{\min}}^{\infty} dM_1 \frac{\partial n}{\partial M_1} \int_{M_{\min}}^{\infty} dM_2 \frac{\partial n}{\partial M_2} \xi_H(\mathbf{x}_1 - \mathbf{x}_2 | M_1, M_2)}{\int_{M_{\min}}^{\infty} dM_1 \frac{\partial n}{\partial M_1} \int_{M_{\min}}^{\infty} dM_2 \frac{\partial n}{\partial M_2}}$$

where we have weighted by the explicit mass profile through the mass function. We can simplify, as

$$\xi_H(\mathbf{x}_1 - \mathbf{x}_2 | M_{\min}) = \frac{\int_{M_{\min}}^{\infty} dM_1 \frac{\partial n}{\partial M_1} b(M_1) \int_{M_{\min}}^{\infty} dM_2 \frac{\partial n}{\partial M_2} b(M_2)}{\int_{M_{\min}}^{\infty} dM_1 \frac{\partial n}{\partial M_1} \int_{M_{\min}}^{\infty} dM_2 \frac{\partial n}{\partial M_2}} \xi_T(\mathbf{x}_1 - \mathbf{x}_2) = \beta^2(M_{\min}) \xi_T(\mathbf{x}_1 - \mathbf{x}_2)$$

where

$$\beta(M_{\min}) = \frac{\int_{M_{\min}}^{\infty} dM \frac{\partial n}{\partial M} b(M)}{\int_{M_{\min}}^{\infty} dM \frac{\partial n}{\partial M}}$$

The physical picture is that clustering is excessive in regions of overdensity, and voids remain greatly deprived of halos. The bias factors the matter tracer densities over the halo density, which provokes higher correlation at greater masses.

Gravitational Wave Events

Rates and Numericals

We compute the number of mergers per chirp mass $d\mathcal{M}$, per unit stellar mass dM_* , given by

$$\frac{dN_{\text{merge}}(M, z)}{d\mathcal{M} dM_*}$$

Noting the merger formation rate $\Psi = \dot{M}_*$, we have the number of mergers per $d\mathcal{M}$, per unit time, as

$$\Psi \frac{dN_{\text{merge}}(M, z)}{d\mathcal{M} dM_*}$$

Thereby, the number of mergers per $d\mathcal{M}$, per unit time, integrating over the halo mass rate, per unit comoving volume, in the source frame,

$$\int d\Psi \Psi \frac{dN_{\text{merge}}(M, z)}{d\mathcal{M} dM_*} \frac{\partial n(M, z)}{\partial M} \frac{\partial M}{\partial \Psi}$$

We now, convert from redshift to the comoving volume, $\frac{dV}{dz} = 4\pi\chi^2(z) \frac{d\chi}{dz}$, where $\frac{d\chi}{dz} = \frac{c}{a(z)}$, is the transformation. Accounting for time dilation due to gravitational redshift, we have, the

transferred rate of mergers per unit $d\mathcal{M}$ in observer's frame per dz since we have converted from comoving volume, to arrive at

$$\frac{1}{1+z} \frac{d\mathcal{V}}{dz} \int d\Psi \Psi \frac{dN_{\text{merge}}(M, z)}{d\mathcal{M}dM_*} \frac{\partial n(M, z)}{\partial M} \frac{\partial M}{\partial \Psi}$$

Further crucial considerations are to include the delay time between the formation of a star, due to the age of a star, formation of a *common envelope*, both of which are minute compared to cosmic scales, and the effective merger time scale, that can be extensively long. Accounting for the time delay due to a probability distribution,

$$\frac{dN_{\text{merge}}}{dt_{\text{obs}}d\mathcal{M}dz} = \frac{1}{1+z} \frac{d\mathcal{V}}{dz} \int d\tau_{\text{delay}} p(\tau_{\text{delay}}) \int dM \Psi \frac{dN_{\text{merge}}(M, z)}{d\mathcal{M}dM_*} \frac{\partial n(M, z)}{\partial M}$$

where we have parametrically accounted for the time delay function.

Further, for clustering, we have,

$$\frac{dN_{\text{merge}}}{dt_{\text{obs}}d\mathcal{M}dzdM} = \frac{1}{1+z} \frac{d\mathcal{V}}{dz} \int d\tau_{\text{delay}} p(\tau_{\text{delay}}) \Psi(M, z) \frac{dN_{\text{merge}}(M, z)}{d\mathcal{M}dM_*} \frac{\partial n(M, z)}{\partial M}$$

where we account for the bias with respect to given mass M , as

$$b_{\text{GW}}(z) = \frac{\int dM b(M, z) \frac{dN_{\text{merge}}}{dt_{\text{obs}}d\mathcal{M}dzdM}}{\int dM \frac{dN_{\text{merge}}}{dt_{\text{obs}}d\mathcal{M}dzdM}}$$

where we have weighted the bias due to halos, with the merger rate per unit chirp mass, per unit observation time, in a range of redshift, for a halo mass bin.

L04.01 Probing Large Scale Structure

Gravitational Waves

Rates

Recap, we have the rate of mergers,

$$\frac{dN_{\text{merge}}}{dt_{\text{obs}} d\mathcal{M} dz dM} = \frac{1}{1+z} \frac{d\mathcal{V}}{dz} \int d\tau_{\text{delay}} p(\tau_{\text{delay}}) \Psi(M, z) \frac{dN_{\text{merge}}(M, z)}{d\mathcal{M} dM_*} \frac{\partial n(M, z)}{\partial M}$$

where we must further select for SNR, the differences in redshift, timescales of halo formation and star formation, and neglecting halo merger.

Neglecting these assumptions, we have the bias with respect to given mass M , as

$$b_{\text{GW}}(z) = \frac{\int dM b(M, z) \frac{dN_{\text{merge}}}{dt_{\text{obs}} d\mathcal{M} dz dM}}{\int dM \frac{dN_{\text{merge}}}{dt_{\text{obs}} d\mathcal{M} dz dM}}$$

where we have weighted the bias due to halos, with the merger rate per unit chirp mass, per unit observation time, in a range of redshift, for a halo mass bin.

Lensing

We must account for magnification bias, shear due to lensing. We have the effective gravitational wave density spike due to density,

$$\delta_{\text{GW}}^{\text{obs}} \approx \delta_{\text{GW}} + (5s - 2) \underbrace{\int_0^{z_{\text{Source}}} dz K(z_{\text{Source}}, z) \delta_{\text{T}}}_{\mathcal{K}}$$

where we have the *convergence* kernel $\mathcal{K}$, with the comoving source redshift z_{Source} , and the slope of the observed halo density s .

The power spectrum and correlation functions are additionally smoothened, due to smear in the angular coordinates due to lensing.

Additional Effects

The presence of peculiar velocities affect the redshift of an object and hence the determination of distances using the standard Hubble-Lemaitre law. The relation between comoving distance

and the cosmological redshift z_{Cosmo} , is

$$\chi(z_{\text{Cosmo}}) = \int_0^{z_{\text{obs}}} dz \frac{c}{H(z)}$$

and the redshift arising from peculiar velocity is

$$z_p = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1$$

where v_r is the radial component of the peculiar velocity. The total redshift is

$$1 + z_{\text{obs}} = (1 + z_{\text{Cosmo}})(1 + z_p)$$

In case of non-relativistic velocities and small redshifts, the relation simplifies to

$$z_{\text{obs}} \approx \frac{H_0 r}{c} + \frac{v_r}{c}$$

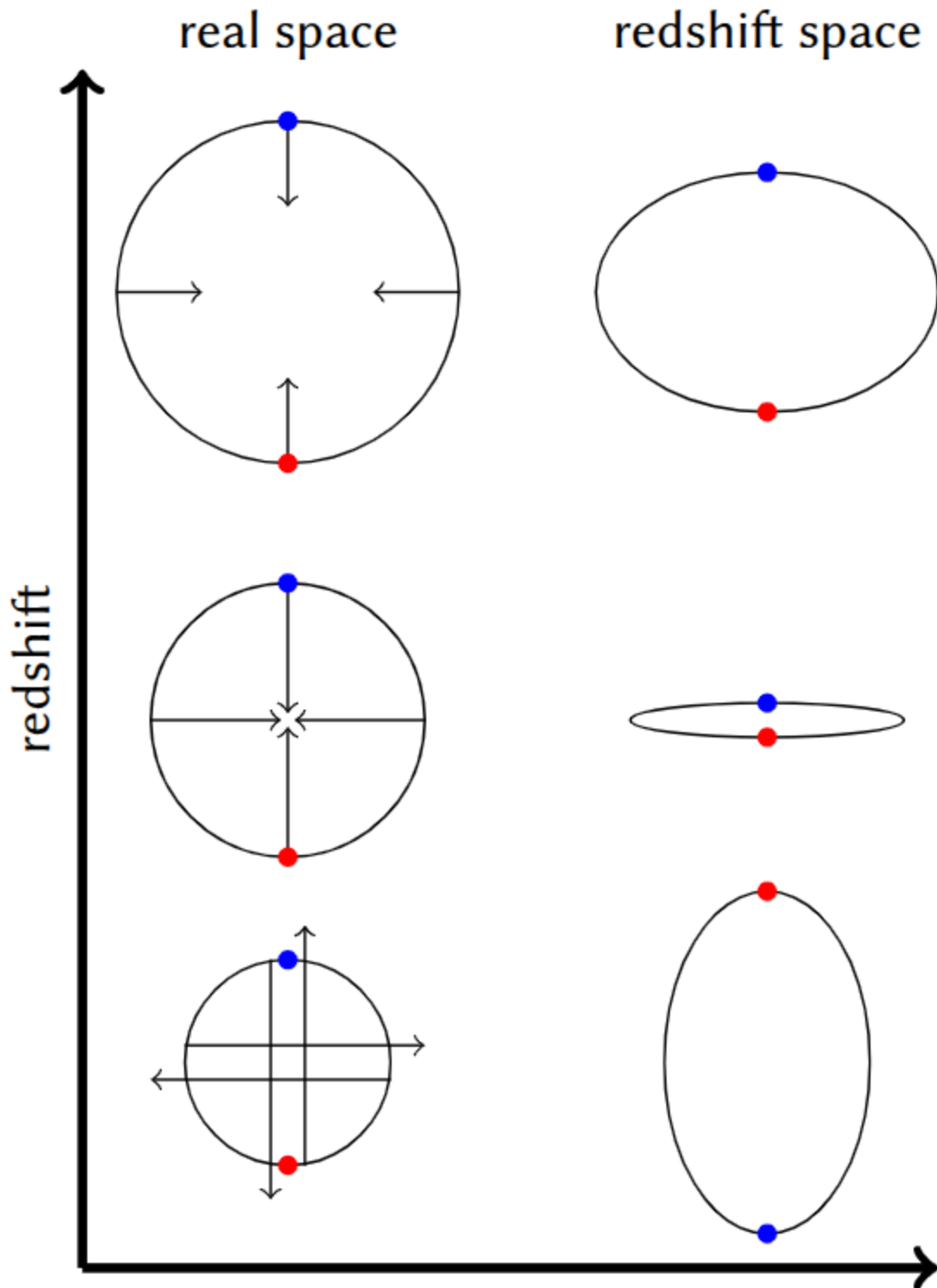
which is valid only in the nearby universe and has to be modified for high redshifts.

We further have the notion of the *redshift distance*, related to the physical distance, by

$$\mathbf{s} = \mathbf{r} + \frac{\mathbf{v} \cdot \hat{\mathbf{r}}}{H_0} \hat{\mathbf{r}}$$

which is derived from the Hubble-Lemaître law. The object will seem nearer in the redshift space when it is moving towards us compared to the Hubble expansion.

We have the effects of peculiar velocities, with large scale structures compressed in redshifted space, while small scale structures are elongated, in a *finger-of-God* effect.



For a large radius within which the overdensity is small, the expansion of the mass shell is decelerated but its peculiar velocity is still too small to compensate for the Hubble expansion. In redshift space the mass shell will then appear squashed along the line-of-sight when observed from a distance much larger than its size.

A mass shell with linear overdensity $\delta \sim 1$ is just turning around at the time it is observed, so its peculiar infall velocity is exactly equal to the Hubble expansion velocity across its radius. In redshift space this shell appears completely collapsed to an observer at large distance.

A mass shell which has already turned around has a peculiar infall velocity which exceeds the Hubble expansion across its radius. If this infall velocity is less than twice the Hubble expansion velocity, the shell appears flattened along the line-of-sight, but with the nearer side having larger redshift distance than the farther side.

At smaller radii the peculiar infall velocities of collapsing shells are much larger than the relevant Hubble velocities and are randomised by scattering effects. The structure then appears to be elongated along the line-of-sight in redshift space (a "finger-of-God" pointing back to the observer).

Density Contrast in the Redshift Space

It is usually the case that the scale of perturbations is much smaller than the distance from us. In that case one can use the plane parallel approximation and construct a local Cartesian coordinate system. We have,

$$\mathbf{s} = \mathbf{r} + \frac{v_z}{H_0} \hat{\mathbf{z}}$$

Using the conservation of mass (or equivalently the conservation of galaxy counts), we can write,

$$\varrho_{\text{T}}^{(s)}(\mathbf{s}) d^3 s = \varrho_{\text{T}}(\mathbf{r}) d^3 r$$

Hence, the density contrasts, relate as,

$$[1 + \delta_{\text{T}}^{(s)}(\mathbf{s})] d^3 s = [1 + \delta_{\text{T}}(\mathbf{r})] d^3 r$$

We relate the coordinate transformation by the Jacobian in the z component,

$$\left| \frac{\partial s^\alpha}{\partial r^\beta} \right| = 1 + \frac{1}{H_0} \frac{\partial v_z}{\partial r_z}$$

Hence,

$$[1 + \delta_{\text{T}}^{(s)}(\mathbf{s})] = [1 + \delta_{\text{T}}(\mathbf{r})] \left(1 + \frac{1}{H_0} \frac{\partial v_z}{\partial r_z} \right)^{-1}$$

For the linear approximation, in first order,

$$\delta_{\text{T}}^{(s)}(\mathbf{s}) \approx \delta_{\text{T}}(\mathbf{r}) - \frac{1}{H_0} \frac{\partial v_z}{\partial r_z}$$

We can now work in the Fourier space, to arrive at

$$\begin{aligned}\delta_{\text{T}}^{(s)}(\mathbf{k}) &\approx \delta_{\text{T}}(\mathbf{k}) - \frac{ik_z}{H_0}v_z(\mathbf{k}) \\ &\approx \delta_{\text{T}}(\mathbf{k}) \left(1 + \beta \frac{k_z^2}{k^2}\right)\end{aligned}$$

where β relates to the bias weighted by the form function. Define $\mu_{\mathbf{k}} = \frac{k_z^2}{k^2}$, such that $\cos \theta = \mu_{\mathbf{k}}$ where θ is the angle between $\mathbf{k}$ and the line of sight. Thereby, we have,

$$\delta_{\text{T}}^{(s)}(\mathbf{k}) \approx \delta_{\text{T}}(\mathbf{k})[1 + \beta\mu_{\mathbf{k}}^2]$$

which relates the density contrasts in the real and redshift spaces. The power spectra are related, as

$$P^{(s)}(\mathbf{k}) \approx P(\mathbf{k})[1 + \beta\mu_{\mathbf{k}}^2]^2$$

showing that the effect of redshift space is to make the power spectrum anisotropic. This effect at large scales is the *Kaiser* effect.