Non-Circular Encounters in Ringdowns

Presentation

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Agenda

Newtonian Effective Potential

[Pakiela, Steven, et al., 2024]

$$\tilde{V}_{\text{eff}} = -\frac{M}{r} + \frac{\tilde{l}^2}{r^2}$$

- Circular orbits at the global minimum, where $\frac{\partial \tilde{V}}{\partial r} = 0$.
- For bound orbits, $\tilde{E} < 0$ but larger than the minimum of the potential: elliptical with $r_{\pm} = -\frac{M}{2\tilde{E}}\left(1 \pm \sqrt{\frac{2\tilde{E}\tilde{l}^2}{M^2} + 1}\right)$.
- Parabolic for $\tilde{E} = 0$, Hyperbolic for $\tilde{E} > 0$.

Schwarzschild Effective Potential

[Pakiela, Steven, et al., 2024]

$$\begin{aligned}\tilde{V}_{\text{eff}} &= -\frac{M}{r} + \frac{\tilde{l}^2}{r^2} - \frac{M\tilde{l}^2}{r^3} \\ &= -\frac{M}{r} + \frac{\tilde{l}^2}{2r^2}\left(1 - \frac{R_s}{r}\right)\end{aligned}$$

Additional term negligible at large radii. Secondary extremum in potential.

Effective One Body Framework

[Buonanno, Alessandra, and Thibault Damour, 1999]

- Two-body dynamics $\rightarrow$ One-effective body in deformed spactime, parameter $\nu = \frac{\mu}{M} = \frac{m_1 m_2}{M^2}$
- Classical $E_{\text{real}}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \left(\frac{E_{\text{eff}}}{\mu}\right)$ formalism
- $H_{\text{Newt}} + \sum H_{\text{PN}} \rightarrow \mu\sqrt{A_\nu(r)\left[1 + \frac{p^2}{\mu^2} + \left(\frac{1}{B_\nu(r)} - 1\right)\right]\frac{p_r^2}{\mu^2}}$
- $ds^2 = -A_\nu(r)dt^2 + B_\nu(r)dr^2 + r^2 d\Omega^2$

Effective One Body (EOB) Framework

[Buonanno, Alessandra, and Thibault Damour, 1999]

EOB Hamiltonian $\hat{H} \equiv \frac{H}{\mu} = \frac{M}{\mu}\sqrt{1 + \frac{2\mu}{M}(H_{\text{eff}} - 1)}$. $H_{\text{eff}} = H_{\text{eff}}^{\text{orb}} + H_{\text{spin-oribt}}$ where $H_{\text{eff}}^{\text{orb}} = \sqrt{A\left(1 + \frac{p_\phi^2}{r_c^2} + Q\right) + p_{r_*}^2}$.

Potential energy $E = M\sqrt{1 + \frac{2\mu}{M}(\hat{W}_{\text{eff}} - 1)}$ where $\hat{W}_{\text{eff}} = \sqrt{A(r)\left(1 + \frac{p_\phi^2}{r^2}\right)}$ defining stability at $\frac{\partial \hat{W}_{\text{eff}}}{\partial r} = \frac{\partial^2 \hat{W}_{\text{eff}}}{\partial r^2} = 0$.

Orbits

[Gamba, Rossella, et al, 2023]

$$E_{\min} \equiv \mu \hat{H}(r_0, q, p_\phi, p_r = 0), \; E_{\max} \equiv \mu \max_r \hat{H}(r, q, p_\phi, p_r = 0).$$

- $(E_{\max}, E_{\min})$ correspond to unstable and stable circular orbits.
- $E_0 > E_{\max}$ Head-on collisions.
- $M < E_0 \leq E_{\max}$ direct plunge, close passages, zoom-whirl.

Qualitative arguments remain unchanged with radiation reaction.

Configuration Dynamics

[Gamba, Rossella, et al, 2023]

$\Omega(t) = \dot{\varphi}$ peaks correspond to Periastron passage

- $E_0 > E_{\mathrm{max}}$ Direct capture $\rightarrow$ Ringdown phase.
- $E < E_{\mathrm{max}}$ Scattering scenario with various encounters.

Orbital Evolution

[Ori, Amos, and Kip S. Thorne, 2000, Buonanno, Alessandra, and Thibault Damour, 2000]

- The *adiabatic inspiral* regime, in which the body gradually descends through a sequence of geodesic orbits with gradually changing constants of the motion. Loss of energy and angular momentum.
- A *transition regime*, in which the character of the orbit gradually changes from inspiral to plunge. Quasi-circular motion, with the ratio of the energy radiated to angular momentum radiated equal to the orbital angular velocity. Gradually changing the effective potential for radial geodesic motion.
- A *plunge* regime, in which the body plunges into the horizon along a geodesic with (nearly) unchanging energy, angular momentum. The effective potential has become so steep that its inward force on the body dominates strongly over radiation reaction.

Ringdown Structure

[Carullo, Gregorio, et al., 2024]

- From initial $(E_0^{\text{ADM}}, J_0^{\text{ADM}})$, at time t, $(E(t), J(t))$, such that
  $E(t) = E_0^{\text{ADM}} - \int_{t_0}^{t} dt' \dot{E}(t')$ and $J(t) = J_0^{\text{ADM}} - \int_{t_0}^{t} dt' \dot{J}(t')$, where
  $\dot{E} = \frac{1}{16\pi} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} |\dot{h}_{\ell m}|^2$ and $\dot{J} = \frac{1}{16\pi} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} m \Im\{h_{\ell m} \dot{h}^*_{\ell m}\}$.
- Merger quantities at t_{mrg} corresponding to the peak of the emission immediately before a ringdown begins.

Quasi-Normal Modes

[Berti, Emanuele, Vitor Cardoso, and Clifford M. Will., 2006]

Perturbed Kerr Black Hole

$$h_+ - ih_\times = -\frac{2}{r^4}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2}\sum_{\ell,m} S_{\ell m}(\iota,\beta)R_{\ell m\omega}(r),$$

Radial Teukolsky function $R_{\ell m\omega} \to r^3 \mathcal{Z}^{\text{out}}_{\ell mn}\exp(-i\omega r)$ as $r \to \infty$ with $\mathcal{Z}^{\text{out}}_{\ell mn}$ being a complex amplitude. Spin-2 Spin-weighted Spheroidal harmonics $S_{\ell m}(\iota,\beta)$.

Quasi-Normal Modes

[Berti, Emanuele, Vitor Cardoso, and Clifford M. Will., 2006]

Exponentially decaying sinusoids given by

$$h_+ - ih_\times = \frac{1}{r}\sum_{\ell mn} \exp(i\omega_{\ell mn}t)\exp(-t/\tau)S_{\ell mn}(\iota,\beta)\mathcal{Z}^{\text{out}}_{\ell mn},$$

Factoring out $\mathcal{Z}^{\text{out}}_{\ell mn} = M\mathcal{A}_{\ell mn}\exp(i\Phi_{\ell mn})$, we have

$$h_+ - ih_\times = \frac{M}{r}\sum_{\ell mn} \mathcal{A}_{\ell mn}\exp(i\omega_{\ell mn}t + \Phi_{\ell mn})\exp(-t/\tau)S_{\ell mn}.$$

Ringdown Waveform model

[Carullo, Gregorio., 2024]

Formally, Superposition of damped sinusoids and an exponentially decaying tail term.

$$h_{\ell m}(t) = \sum_{\ell' mn}^{\infty} \Big[A^{+}_{\ell n} \exp\{i(\omega^{+}_{\ell' mn}(t - t_{\text{ref}}) + \phi^{+}_{\ell' mn})\} + \\ A^{-}_{\ell' mn} \exp\{i(\omega^{-}_{\ell' mn}(t - t_{\text{ref}}) + \phi^{-}_{\ell' mn})\} \Big] \theta(t - t^{\text{ref}}_{\text{start}}) + \\ A^{T}_{\ell m} \exp\{i\phi^{T}_{\ell m}\}(t - t_{\text{ref}})^{p^{T}_{\ell m}} \theta(t - t^{\text{tail}}_{\text{start}}).$$

Waveform Features

[Carullo, Gregorio., 2024]

- Spherical-spheroidal mode mixing. Results in infinite summation over modes ℓ.
- Co(counter)-rotating modes have $\Re\left\{\omega^{\pm}_{\ell mn}\right\} > (<) 0$.
- Orbital angular momentum alignment, unless extreme intrinsic spins.
- Time-independent tail terms considered. Constant fitting to remove simulation inaccuracies.
- $t_{\text{ref}} = t_{\text{mrg}}$, last peak of A_{22} immediately before a ringdown regime begins.

Earlier Work

[Damour, Thibault, and Alessandro Nagar, 2014]

- Identify the mass M_{BH} and angular momentum J_{BH} of the final black hole using the prediction of the EOB dynamics or using Numerical Relativity (NR) fitting formulas.
- Using $(M_{\text{BH}}, J_{\text{BH}})$ to compute a set of quasi-normal mode (QNM) frequencies and to build a linear superposition of QNMs with coefficients determined by imposing continuity at a certain merger time $t = t_0$ determined from the EOB dynamics.

Analytic Representation

[Damour, Thibault, and Alessandro Nagar, 2014]

- Defining the QNM-rescaled waveform as $\bar{h}(\tau) = \exp(\sigma_1 \tau + \imath \phi_{22}^{\text{mrg}}) h(\tau)$, where we factor out the contribution of the fundamental QNM $h_1(\tau) \sim \exp(-\sigma_1 \tau)$ with $\sigma_1 = \alpha_1 + \imath \omega_1$, and ϕ_{22}^{mrg} being the value of ϕ_{22} at merger.
- Decomposing $\bar{h}(\tau) \equiv A_{\bar{h}}(\tau) \exp(\imath \phi_{\bar{h}}(\tau))$ with $\phi_{\bar{h}}(\tau) = \omega_1 \tau - \phi_{22}(\tau) + \phi_{22}^{\text{mrg}}$.

Analytic Representation

[Damour, Thibault, and Alessandro Nagar, 2014]

$$A_{\bar{h}}(\tau) = c_1^A \tanh(c_2^A \tau + c_3^A) + c_4^A$$

$$\phi_{\bar{h}}(\tau) = -c_1^\phi \ln\left(\frac{1 + c_3^\phi \exp(-c_2^\phi \tau) + c_4^\phi \exp(-2c_2^\phi \tau)}{1 + c_3^\phi + c_4^\phi}\right)$$

- I $A_{\bar{h}}(\tau = 0) = A_{22}^{\mathsf{mrg}}$ (NR).
- II $\left.\frac{dA_{\bar{h}}}{d\tau}\right|_{\tau=0} = \dot{A}_{22}\big|_{t=t_0}$.
- III $c_2^A = \frac{\alpha_2 - \alpha_1}{2}$.
- IV $\left.\frac{d\phi_{\bar{h}}}{d\tau}\right|_{\tau=0} = \Delta\omega = \omega_1 - \mathcal{M}_{\mathsf{BH}}\omega_{22}^{\mathsf{mrg}}$.
- V $c_2^\phi = \alpha_2 - \alpha_1$.

Analytic Representation

[Damour, Thibault, and Alessandro Nagar, 2014]

From the continuity conditions,

- $c_1^A = A_{22}^{\mathsf{mrg}} \alpha_1 \frac{\cosh^2 c_3^A}{c_2^A}$ (from II, III)
- $c_2^A = \frac{\alpha_2 - \alpha_1}{2}$ (from III)
- c_3^A *free parameter*
- $c_4^A = A_{22}^{\mathsf{mrg}} - c_1^A \tanh(c_3^A)$ (from I)

- $c_1^\phi = \frac{(\omega_1 - \mathcal{M}_{\mathsf{BH}} \omega_{22}^{\mathsf{mrg}})(1 + c_3^\phi + c_4^\phi)}{c_2^\phi (c_3^\phi + 2 c_4^\phi)}$ (from IV)
- $c_2^\phi = \alpha_2 - \alpha_1$ (from IV)
- c_3^ϕ *free parameter*
- c_4^ϕ *free parameter*

EOB Parameterization

[Carullo, Gregorio, et al., 2024]

- Mass-normalized energy $h \equiv \frac{E}{M}$, and angular momentum $j \equiv \frac{J}{\nu M^2}$.
- EOB Effective energy $\hat{E}_{\text{eff}} = \frac{E_{\text{eff}}}{\mu} = 1 + \frac{h^2-1}{2\nu}$, and impact parameter $\hat{b}_{\text{EOB}} = \frac{b_{\text{EOB}}}{M} = \frac{jh}{\sqrt{\hat{E}_{\text{eff}}^2-1}}$. For bound configurations with $\hat{E}_{\text{eff}} < 1$, $\hat{b}_{\text{EOB}} = \frac{jh}{E_{\text{eff}}}$ at merger as dynamical impact parameter.

Effective Characterization

[Carullo, Gregorio, et al., 2024, Carullo, Gregorio., 2024]

- Non-bijective amplitude dependence on eccentricity. Gauge-invariant parameterisation, shifting the perspective from orbital-based parameterisations to dynamics-based ones.
- Effective parameter space of quasi-universality through derived impact parameter at merger, the effective energy and the perturbation parameter of the metric $(\nu, \hat{b}_{\text{mrg}}, E_{\text{eff}}^{\text{mrg}})$ for the global fit.
- Unwrapped phases $\hat{\phi}_{\ell mn} \longrightarrow \hat{\phi}_{\ell mn} + 2n\pi$ with $n \in \mathbb{Z}$.

Impact parameter

[Carullo, Gregorio, et al., 2024, Carullo, Gregorio., 2024]

- Effective dynamics as eccentricity e_0 increases results in the merger, and the periastron radius moves closer to the last stable orbit radius. Smooth blending into the merger.
- For medium values of e_0, a large burst of GWs is emitted on the last periastron just before the merger, implying a large loss of energy from the system, hence a reduced merger-ringdown amplitude.
- Initial increase in merger amplitude over the transition since less energy has been lost in previous encounters and due to an inversion point of the orbit with maximum emission.

Extracting Amplitudes and Phases

[Forteza, Xisco Jiménez, et al., 2023, Carullo, Gregorio., 2024]

- Ratios (differences) of amplitudes (phases) to the quasi-circular limit. Eliminates secondary dependencies, and we obtain factorized (additive) form for the fits.
- $\hat{A}_{\ell mn} \longrightarrow \frac{A_{\ell mn}}{A^{qc}_{\ell mn}}$ and $\hat{\phi}_{\ell mn} \longrightarrow 2\phi_{\ell mn} - m\phi_{220} - \phi^{qc}_{\ell mn}$. Additional subtraction of the $m\phi_{220}$ factor to eliminate the dependence on the arbitrary phase of the specific simulation.

Rational Fit

[Carullo, Gregorio, et al., 2024, Carullo, Gregorio., 2024]

$$\tilde{Y} = \prod_{i=1}^{N} \tilde{Y}_0 \left(\frac{1 + p_{1,i} Q_i + p_{2,i} Q_i^2}{1 + p_{3,i} Q_i + p_{4,i} Q_i^2} \right)$$

- We associate the fit for $\hat{A}_{\ell mn}$ and $\phi_{\ell mn}$ based on quasi-circular results.
- $(Y_0, p_{k,i}) \in \mathbb{R}$ with $p_{k,i} = b_{k,i}(1 + c_{k,i} X)$ where $X = 1 - 4\nu$ and $b_{k,i}, c_{k,i} \in \mathbb{R}$ for $N = 2$. Here, i runs over the N fitting variables, with $Q_i \in \{\nu, \hat{b}_{\mathrm{mrg}}, E_{\mathrm{eff}}^{\mathrm{mrg}}\}$ with $N = 3$.

Newer Representation

[Albanesi, Simone, et al., 2023]

$$A_{\bar{h}}(\tau) = \left(\frac{c_1^A}{1 + \exp(-c_2^A \tau + c_3^A)} + c_4^A \right)^{\frac{1}{c_5^A}}$$

$$\phi_{\bar{h}}(\tau) = -c_1^\phi \ln \left(\frac{1 + c_3^\phi \exp(-c_2^\phi \tau) + c_4^\phi \exp(-2c_2^\phi \tau)}{1 + c_3^\phi + c_4^\phi} \right)$$

1. $A_{\bar{h}}(\tau = 0) = A_{22}^{\mathrm{mrg}}$ (NR).
2. $\left.\frac{dA_{\bar{h}}}{d\tau}\right|_{\tau=0} = \dot{A}_{22}\big|_{t=t_0}$.
3. $\left.\frac{d^2 A_{\bar{h}}}{d\tau^2}\right|_{\tau=0} = \ddot{A}_{22}\big|_{t=t_0}$.
4. $\left.\frac{d\phi_{\bar{h}}}{d\tau}\right|_{\tau=0} = \Delta\omega = \omega_1 - \mathcal{M}_{\mathrm{BH}}\omega_{22}^{\mathrm{mrg}}$.

Newer Representation

[Albanesi, Simone, et al., 2023]

From the continuity conditions,

- $c_1^A = \frac{c_5^A \alpha_1}{c_2^A} (A_{22}^{\text{mrg}})^{c_5^A} \exp(-c_3^A)(1+\exp(c_3^A))^2$ (from II)
- c_2^A *free parameter*
- c_3^A *free parameter*
- $c_4^A = (A_{22}^{\text{mrg}})^{c_5^A} - \frac{c_1^A}{1+\exp(c_3^A)}$ (from I)
- $c_5^A = -\frac{\ddot{A}_{22}^{\text{mrg}}}{A_{22}^{\text{mrg}} \alpha_1^2} + \frac{c_2^A}{\alpha_1} \frac{\exp(c_3^A)-1}{\exp(c_3^A)+1}$ (from IV)

- $c_1^\phi = \frac{1+c_3^\phi+c_4^\phi}{c_2^\phi(c_3^\phi+2c_4^\phi)} \Delta\omega$ (from IV)
- c_2^ϕ *free parameter*
- c_3^ϕ *free parameter*
- c_4^ϕ *free parameter*

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Thank You!

Questions? Comments?