Isolated Black Hole Horizons

Master's Thesis

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Certificate

This is to certify that this dissertation, entitled *Isolated Black Hole Horizons* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Nishkal Rao at Indian Institute of Science Education and Research under the supervision of Dr. Badri Krishnan, Radboud University, during the academic year 2025-26.

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Dedicated to my parents, Meera and Athul,
And to all other parents who struggle
So their children don't have to.

Declaration

I hereby declare that the matter embodied in the report entitled *Isolated $\mathscr{B}$lack Hole Horizons* are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education & Research (IISER) Pune, under the supervision of Dr. Badri Krishnan, and the same has not been submitted elsewhere for any other degree. Wherever others contribute, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

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Abstract

Acknowledgments

Part I

Introduction

In some strange way, any new fact or insight that I may have found has not seemed to me as a "discovery" of mine, but rather something that had always been there and that I had chanced to pick up.

Subrahmanyan Chandrasekhar

Chapter 1

Primer on General Relativity

1.1 Foundations

We consider a spacetime $(\mathscr{M}, \boldsymbol{g})$ where $\mathscr{M}$ is a $\mathcal{C}^\infty$ manifold of dimension n, and $\boldsymbol{g}$ is a Lorentzian metric on $\mathscr{M}$, of signature $(-,+,+,+)$. Let ∇ be the affine connection associated with $\boldsymbol{g}$ on $\mathscr{M}$.

At any point $p \in \mathscr{M}$, we denote $T_p\mathscr{M}$ as the tangent space of vectors at p, whose dual cotangent space $T^*_p\mathscr{M}$ is constituted by all linear forms at p, which map vector fields to the space of smooth scalar fields. Setting $\{\mathbf{e}_\alpha\}_{\alpha=1}^n$ as a vector basis of the tangent space $T_p\mathscr{M}$, we can associate a dual basis $\{\mathbf{e}^\alpha\}_{\alpha=1}^n$ for $T^*_p\mathscr{M}$, such that $\mathbf{e}^\alpha\left(\mathbf{e}_\beta\right) = \delta^\alpha_\beta$. The metric $\boldsymbol{g}$ introduces a natural isomorphism between $T_p\mathscr{M}$ and $T^*_p\mathscr{M}$, which, in index notation, corresponds to lowering or raising the index by contraction.

For a general tensor $\boldsymbol{T}$ of type $\binom{p}{q}$, we can define its components $T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}$ with respect to the basis $\mathbf{e}^\alpha$ and $\mathbf{e}_\alpha$, such that, we have the expansion

$$\boldsymbol{T} = T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}\, \mathbf{e}_{\alpha_1} \otimes \mathbf{e}_{\alpha_2} \otimes \cdots \otimes \mathbf{e}_{\alpha_p} \otimes \mathbf{e}^{\beta_1} \otimes \mathbf{e}^{\beta_2} \otimes \cdots \mathbf{e}^{\beta_q}. \tag{1.1.1}$$

Further, we define the covariant derivative of the tensor field $\nabla\boldsymbol{T}$, with components,

$$\nabla\boldsymbol{T} = \nabla_\gamma T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}\, \mathbf{e}_{\alpha_1} \otimes \mathbf{e}_{\alpha_2} \otimes \cdots \otimes \mathbf{e}_{\alpha_p} \otimes \mathbf{e}^{\beta_1} \otimes \mathbf{e}^{\beta_2} \otimes \cdots \mathbf{e}^{\beta_q} \otimes \mathbf{e}^\gamma, \tag{1.1.2a}$$

where we realise the covariant derivative for the tensor $\boldsymbol{T}$ along any vector field $\boldsymbol{v}$ is related to $\nabla\boldsymbol{T}$ by contracting the last 1-form index with the vector, as,

$$\nabla_{\boldsymbol{v}}\boldsymbol{T} = \nabla\boldsymbol{T}(\underbrace{\dots,\dots,}_{p+q}\boldsymbol{v}), \tag{1.1.2b}$$

with components given by $v^\gamma\nabla_\gamma T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}$.

Given the tensor field $\boldsymbol{T}$, we can define the *Lie derivative*, $\mathcal{L}_{\boldsymbol{v}}\boldsymbol{T}$ of $\boldsymbol{T}$ with respect to a vector field $\boldsymbol{v}$, as the infinitesimal change along the flow of $\boldsymbol{v}$, through its components, as

$$(\mathcal{L}_{\boldsymbol{v}}\boldsymbol{T})^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q} = v^\gamma\nabla_\gamma T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q} - \sum_{i=1}^{p} T^{\alpha_1\dots\gamma\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}\nabla_\gamma v^{\alpha_i} - \sum_{i=1}^{q} T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\dots\gamma\dots\beta_q}\nabla_{\beta_i} v^\gamma, \tag{1.1.2c}$$

which can be seen to be independent of the connection involved in the definition of the covariant derivative, and depends solely on the differentiable structure of the manifold $\mathscr{M}$, and not upon the metric $\boldsymbol{g}$, or any affine connection ∇.

The *Riemann curvature tensor*, $\boldsymbol{\mathcal{R}}$ of the spacetime connection ∇, is defined through its action on the space $T_p^*\mathscr{M} \times T_p^{\times 3}\mathscr{M}$ as, for $\boldsymbol{\omega} \in T_p^*\mathscr{M}$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ each in $T_p\mathscr{M}$, are mapped to $\mathcal{C}^\infty(\mathscr{M}, \mathbb{R})$ through,

$$\boldsymbol{\mathcal{R}}(\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}\left(\boldsymbol{\omega}, \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}\right), \tag{1.1.3a}$$

which can be explicitly defined through its components, $R_{\alpha\beta\gamma\delta}$, given by the Ricci identity as,

$$\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)w^\gamma = R^\gamma_{\mu\alpha\beta}w^\mu. \tag{1.1.3b}$$

The Riemann tensor is clearly antisymmetric in the last two arguments, $(\boldsymbol{u}, \boldsymbol{v})$. We can gain an additional relation employed through the compatibility of metric $\boldsymbol{g}$, with the affine connection ∇, since we have $\boldsymbol{\mathcal{R}}(\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = -\boldsymbol{\mathcal{R}}(\underline{\boldsymbol{w}}, \vec{\boldsymbol{\omega}}, \boldsymbol{u}, \boldsymbol{v})$, leading to an antisymmetry, with the corresponding duals, $\underline{\boldsymbol{w}} \in T_p^*\mathscr{M}$ and $\vec{\boldsymbol{\omega}} \in T_p\mathscr{M}$, through the induced isomorphism from $\boldsymbol{g}$. In addition, we have the cyclic property, given by,

$$\boldsymbol{\mathcal{R}}(\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{\mathcal{R}}(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) + \boldsymbol{\mathcal{R}}(\boldsymbol{\omega}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}), \tag{1.1.4}$$

realised through the Bianchi identity for the covariant derivative.

Furthermore, we denote the Ricci tensor $\boldsymbol{R}$, as the symmetric bilinear form, in components, given by,

$$R_{\alpha\beta} = R^\mu_{\alpha\mu\beta}, \tag{1.1.5}$$

which is a contraction of the Riemann tensor.

The Riemann tensor can be formulated into a pure trace part, represented by the Ricci scalar, $R = g^{\alpha\beta}R_{\alpha\beta}$, an additional trace part characteristic of the Ricci tensor, $\boldsymbol{R}$, and a traceless part. The Weyl conformal curvature tensor, $\boldsymbol{C}$, is defined formally by the trace-free part of the Riemann tensors, given, in component form, as

$$\begin{aligned} R^\gamma_{\delta\alpha\beta} = C^\gamma_{\delta\alpha\beta} &+ \frac{1}{2}\left(R^\gamma_\alpha g_{\delta\beta} + R^\gamma_\beta g_{\delta\alpha} - R_{\delta\beta}\delta^\gamma_\alpha - R_{\delta\alpha}\delta^\gamma_\beta\right) \\ &- \frac{1}{6}R\Big(g_{\delta\alpha}\delta^\gamma_\beta - g_{\delta\beta}\delta^\gamma_\alpha\Big). \end{aligned} \tag{1.1.6}$$

This tensor has manifestly all the symmetries of the Riemann tensor, with a subtle difference in the tracing out, given the antisymmetry as $C^\mu_{\alpha\mu\beta} = 0$, in contrast to $R^\mu_{\alpha\mu\beta} = R_{\alpha\beta}$. In four dimensions, it is well known that the 20 independent components of the Riemann tensor are distributed into 10 components in the Ricci tensor and 10 independent components in the Weyl tensor.

We consider a spacetime $(\mathscr{M}, \boldsymbol{g})$ such that $\boldsymbol{g}$ obeys to the Einstein equation (with zero cosmological constant), given by

$$\boldsymbol{R} - \frac{1}{2} R \boldsymbol{g} = 8\pi \boldsymbol{T}, \tag{1.1.7a}$$

where $\boldsymbol{T}$ is the matter stress-energy tensor, which can be equivalently written as,

$$\boldsymbol{R} = 8\pi \left(\boldsymbol{T} - \frac{1}{2} T \boldsymbol{g} \right), \tag{1.1.7b}$$

with $T = g^{\mu\nu} T_{\mu\nu}$ as the trace with respect to $\boldsymbol{g}$.

1.2 Null Hypersurfaces

Consider a smooth hypersurface $\mathscr{H}$, which can be inferred as the image of a submanifold $\mathscr{H}_0$ of the spacetime $(\mathscr{M}, \boldsymbol{g})$, given by a smooth embedding map $\Phi : \mathscr{H}_0 \to \mathscr{M}$, such that $\mathscr{H} = \Phi(\mathscr{H}_0)$. The embedding Φ defines a push-forward mapping Φ_* between $T_p\mathscr{H}$ and $T_p\mathscr{M}$, by carrying along tangent vectors $\boldsymbol{v}$, to curves in $\mathscr{H}$, naturally to tangent vectors $\Phi_*\boldsymbol{v}$ in $\mathscr{M}$. Conversely, the embedding Φ induces a pull-back mapping Φ^* between the linear forms $\boldsymbol{\omega}$ in $T_p^*\mathscr{M}$ to be associated with linear forms $\Phi^*\boldsymbol{\omega}$ in $T_p^*\mathscr{H}$, that map vector fields in $T_p\mathscr{H}$ to scalar fields in $\mathbb{R}$.

1.2.1 Fundamental Properties

We define the *first fundamental form* through the *intrinsic metric* $\boldsymbol{q}$ by the extended pull-back operation on multilinear forms, Φ^*, of the spacetime metric $\boldsymbol{g}$ on the hypersurface $\mathscr{H}$, such that $\boldsymbol{q} = \Phi^*\boldsymbol{g}$, realised through the restriction of the bilinear form at each point $p \in \mathscr{H}$, by

$$\boldsymbol{q}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}), \quad \forall (\boldsymbol{u}, \boldsymbol{v}) \in T_p\mathscr{H} \times T_p\mathscr{H}. \tag{1.2.1}$$

When $\mathscr{H}$ is null, the intrinsic metric $\boldsymbol{q}$ is degenerate. This happens if, and only if, there exists a non-vanishing vector field in $T_p\mathscr{H}$, which is orthogonal to all vector fields in $T_p\mathscr{H}$. An equivalent definition demands any vector field $\boldsymbol{\ell}$ in $T_p\mathscr{M}$ which is normal to $\mathscr{H}$, to be a null vector with respect to the metric $\boldsymbol{g}$. We choose such a future-directed normal $\boldsymbol{\ell}$ in the degenerate direction of the metric such that $\boldsymbol{q}(\boldsymbol{\ell}, \boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in T_p\mathscr{H}$. This necessarily restricts the signature of $\boldsymbol{q}$ to $(0,+,+)$. A prominent property of null hypersurfaces is that their normal vectors are both orthogonal and tangential to them.

The null hypersurface $\mathscr{H}$ is ruled by a family of null generators, and each vector field $\boldsymbol{\ell}$ that is normal to $\mathscr{H}$ is tangent to these null geodesics which obey

$$\nabla_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa_{(\boldsymbol{\ell})}\boldsymbol{\ell}, \tag{1.2.2}$$

where $\kappa_{(\boldsymbol{\ell})}$ defines the acceleration of the non-affinely parameterized integral curves of $\boldsymbol{\ell}$, which are geodesics. By a suitable choice of a renormalization factor $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell}$, we can reduce to the classic geodesic equation $\nabla_{\boldsymbol{\ell}'}\boldsymbol{\ell}' = 0$, such that $\kappa'_{(\boldsymbol{\ell})} = \alpha\left(\kappa_{(\boldsymbol{\ell})} + \nabla_{\boldsymbol{\ell}}\ln\alpha\right)$, which can be nullified by appropriate choice of α.

As for any hypersurface, we can further define the notion of a general shape operator to capture the *extrinsic curvature* of $\mathscr{H}$ in $\mathscr{M}$, through the endomorphism of $T_p\mathscr{H}$, defined by the *Weingarten map* $\boldsymbol{\chi} : T_p\mathscr{H} \to T_p\mathscr{H}$, which associates with each tangent vector $\boldsymbol{v} \in T_p\mathscr{H}$, the variation of the normal $\boldsymbol{\ell}$, along that vector, with respect to the spacetime connection ∇, given as,

$$\boldsymbol{\chi}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}}\boldsymbol{\ell}, \tag{1.2.3}$$

which projects the derivative of $\boldsymbol{\ell}$, along vectors tangent to $\mathscr{H}$. The Weingarten map for null hypersurfaces depends on the specific choice of the normal $\boldsymbol{\ell}$, in contrast with the timelike or spacelike case, where the unit length of the normal fixes it unambiguously. For a rescaling of $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell}$, we have $\boldsymbol{\chi}'(\boldsymbol{v}) = \alpha\boldsymbol{\chi}(\boldsymbol{v}) + (\nabla_{\boldsymbol{v}}\alpha)\boldsymbol{\ell}$.

The fundamental property of the Weingarten map is to be self-adjoint with the metric $\boldsymbol{q}$, such that for tangent vectors $\boldsymbol{u}$, $\boldsymbol{v}$ in $T_p\mathscr{H}$, we have $\boldsymbol{q}(\boldsymbol{u},\boldsymbol{\chi}(\boldsymbol{v})) = \boldsymbol{q}(\boldsymbol{\chi}(\boldsymbol{u}),\boldsymbol{v})$. Further, the non-affinity coefficient $\kappa_{(\boldsymbol{\ell})}$ defined earlier, can be corresponded as the eigenvalue of the Weingarten map, corresponding to the eigenvector $\boldsymbol{\ell}$, since $\boldsymbol{\chi}(\boldsymbol{\ell}) = \nabla_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa_{(\boldsymbol{\ell})}\boldsymbol{\ell}$.

We proceed to define the *second fundamental form* of $\mathscr{H}$ with respect to $\boldsymbol{\ell}$, by

$$\begin{aligned}\boldsymbol{\Theta}(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{u}\cdot\boldsymbol{\chi}(\boldsymbol{v}), \quad \forall(\boldsymbol{u},\boldsymbol{v}) \in T_p\mathscr{H}\times T_p\mathscr{H} && (1.2.4\text{a})\\ &= \boldsymbol{u}\cdot\nabla_{\boldsymbol{v}}\boldsymbol{\ell} = \boldsymbol{q}(\boldsymbol{u},\nabla_{\boldsymbol{v}}\underline{\boldsymbol{\ell}})\\ &= \nabla\underline{\boldsymbol{\ell}}(\boldsymbol{u},\boldsymbol{v}), && (1.2.4\text{b})\end{aligned}$$

which is a symmetric bilinear form, due to the self-adjointness of $\boldsymbol{\chi}$. Although $\boldsymbol{\Theta}$ could be defined for any vector field $\boldsymbol{\ell}$, the notion of symmetric form, is due to it being hypersurface orthogonal, which results in the self-adjointness of $\boldsymbol{\chi}$. From the final expression, it can be seen that the $\boldsymbol{\Theta}$ is identically, the pull-back of the bilinear form $\nabla\underline{\boldsymbol{\ell}}$ (not necessarily symmetric) onto $T_p\mathscr{H}$, induced by the embedding Φ of $\mathscr{H}$ in $\mathscr{M}$, given as $\boldsymbol{\Theta} = \Phi^*\nabla\underline{\boldsymbol{\ell}}$, which is symmetric as a consequence of $\boldsymbol{\ell}$ is normal to the hypersurface.

Further, the bilinear form is degenerate in the null direction along $\boldsymbol{\ell}$, similar to the first fundamental form $\boldsymbol{q}$, since $\boldsymbol{\Theta}(\boldsymbol{\ell},\boldsymbol{v}) = \boldsymbol{v}\cdot\boldsymbol{\chi}(\boldsymbol{\ell}) = \kappa_{(\boldsymbol{\ell})}\boldsymbol{v}\cdot\boldsymbol{\ell} = 0$. Further, the dependency on the rescaling of the normal, $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell}$, we have $\boldsymbol{\Theta}' = \alpha\boldsymbol{\Theta}$.

1.2.2 Cross Section

For a null hypersurface, there is no canonical mapping from vectors of $\mathscr{M}$ to $\mathscr{H}$, since there is no natural direction transverse to $\mathscr{H}$, as in vectors belonging to $T_p\mathscr{M} \setminus T_p\mathscr{H}$. Thereby, we define additional structure to orthogonally project naturally along $\boldsymbol{\ell}$ onto the hypersurface $\mathscr{H}$.

We formally define the *cross section* of $\mathscr{H}$, as a submanifold $\mathscr{N}$ of $\mathscr{H}$ of codimension 2, such that the null normal $\boldsymbol{\ell}$ is nowhere tangent to $\mathscr{N}$, and each null geodesic generator of $\mathscr{H}$ intersects $\mathscr{N}$ almost once. For a null hypersurface, it can be realised that any cross section is spacelike, by noting that every nonzero tangent vector $\boldsymbol{v}$ is tangent to a null geodesic generator, that is, normal to the hypersurface.

We can similarly denote the *induced metric* $\boldsymbol{h}$ by the metric $\boldsymbol{g}$ on a cross section $\mathscr{N}$, through the restriction at each point $p \in \mathscr{N}$ by

$$\boldsymbol{h}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}), \quad \forall(\boldsymbol{u},\boldsymbol{v}) \in T_p\mathscr{N} \times T_p\mathscr{N}, \tag{1.2.5}$$

where we realise that the pull-back of $\boldsymbol{h}$ on $\mathscr{H}$, coincides with the pull-back of $\boldsymbol{g}$, that we have denoted $\boldsymbol{q}$ in Eq. (1.2.1).

Further, from the above property that $\mathscr{N}$ is spacelike, it is equivalent that $\boldsymbol{h}$ is positive definite, that is, $\boldsymbol{h}(\boldsymbol{v},\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in T_p\mathscr{H}$, and $\boldsymbol{h}(\boldsymbol{v},\boldsymbol{v}) = 0$ necessarily implies $\boldsymbol{v} = 0$.

At each point p of a cross section $\mathscr{N}$ of a null hypersurface $\mathscr{H}$, the tangent space $T_p\mathscr{N}$ has an *orthogonal complement* $T_p^{\perp}\mathscr{N}$, which is a timelike 2-dimensional vector plane such that the tangent space to $\mathscr{M}$ at p is given by the direct sum

$$T_p\mathscr{M} = T_p\mathscr{N} \oplus T_p^{\perp}\mathscr{N}, \tag{1.2.6}$$

such that every vector in $T_p^{\perp}\mathscr{N}$ is orthogonal to $\mathscr{N}$. This motivates us to define a transverse notion to $\mathscr{H}$, through a local Minkowski null cone at each point. We have two null directions in every null cone, of which $\boldsymbol{\ell}$ is one. Thereby, at each point p of a cross section $\mathscr{N}$ of a null hypersurface $\mathscr{H}$ with future-directed null normal $\boldsymbol{\ell}$, we can choose a future-directed null vector $\boldsymbol{n}$ transverse to $\mathscr{H}$ such that

$$\boldsymbol{\ell} \cdot \boldsymbol{n} = -1, \quad T_p^{\perp}\mathscr{N} = \mathrm{Span}(\boldsymbol{\ell},\boldsymbol{n}), \tag{1.2.7}$$

since $\boldsymbol{\ell}$ and $\boldsymbol{n}$ are non-collinear and the dimension $\dim(T_p^{\perp}\mathscr{N}) = 2$. Note that $\boldsymbol{\ell}$ is independent of the choice of the cross section $\mathscr{N}$ of $\mathscr{H}$ through p, but $\boldsymbol{n}$ does depend on $\mathscr{N}$. This intricacy is closely related to the degeneracy of the induced metric onto the hypersurface, which prevents us from specifying a canonical notion.

From Eq. (1.2.7), we can uniquely decompose any vector orthogonally as $\boldsymbol{u} = \boldsymbol{u}^{\top} + \alpha\boldsymbol{\ell} + \beta\boldsymbol{n}$, where $\boldsymbol{u}^{\top}$ denotes the tangential component of $\boldsymbol{u}$ to $\mathscr{N}$. Thereby, the bilinear form $\boldsymbol{h}$, defined in Eq. (1.2.1), can be extended to all vectors of $T_p\mathscr{M}$ by

$$\boldsymbol{h} = \boldsymbol{g} + \underline{\boldsymbol{\ell}} \otimes \underline{\boldsymbol{n}} + \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{\ell}}, \tag{1.2.8}$$

which clearly obeys $\boldsymbol{h}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{h}(\boldsymbol{u}^\top,\boldsymbol{v}^\top)$, for all $(\boldsymbol{u},\boldsymbol{v}) \in T_p\mathscr{M} \times T_p\mathscr{M}$. Thus, we may then take $\boldsymbol{h}$ as the 4-dimensional extension of $\boldsymbol{q}$, given by the above bilinear form.

We further have the *orthogonal projector* onto the cross section $\mathscr{N}$, defined through the metric dual of the bilinear form $\boldsymbol{h}$ by

$$\vec{\boldsymbol{h}} = \mathbb{I} + \boldsymbol{\ell} \otimes \underline{\boldsymbol{n}} + \boldsymbol{n} \otimes \underline{\boldsymbol{\ell}}, \tag{1.2.9a}$$

which projects the transverse component as $\vec{\boldsymbol{h}}(\boldsymbol{v}) = \boldsymbol{v}^\top$ for all $\boldsymbol{v} \in T_p\mathscr{M}$.

Having introduced the transverse null direction $\boldsymbol{n}$, we can now define the projector $\boldsymbol{\Pi}$ onto $\mathscr{H}$ along $\boldsymbol{n}$ as,

$$\boldsymbol{\Pi}(\boldsymbol{v}) = \boldsymbol{v} + (\boldsymbol{\ell} \cdot \boldsymbol{v})\,\boldsymbol{n}. \tag{1.2.9b}$$

which leaves any vector in $T_p\mathscr{N}$ invariant, and projects vector fields in $T_p\mathscr{M}$ onto $T_p\mathscr{H}$. We note that for any tangent vector $\boldsymbol{v} \in T_p\mathscr{H}$ at a point p, we have $\boldsymbol{\Pi}(\boldsymbol{v}) = \boldsymbol{v}$ since $\boldsymbol{\ell} \cdot \boldsymbol{v} = 0$, while $\boldsymbol{\Pi}(\boldsymbol{n}) = 0$.

We further have the relation to the orthogonal projector dual of the metric $\vec{\boldsymbol{h}}$, with the projector $\boldsymbol{\Pi}$ onto $\mathscr{H}$, through

$$\vec{\boldsymbol{h}}(\boldsymbol{v}) = \boldsymbol{\Pi}(\boldsymbol{v}) + (\boldsymbol{n} \cdot \boldsymbol{v})\,\boldsymbol{\ell}, \tag{1.2.10}$$

which helps us extend the projector from $T_p\mathscr{M}$ onto $T_p\mathscr{H}$. On working through the dual $\boldsymbol{\Pi}^*$ projecting 1-forms in $T^*_p\mathscr{H}$ onto $T^*_p\mathscr{M}$. We obtain a strong duality between $\boldsymbol{\ell}$ and $\boldsymbol{n}$ through $\boldsymbol{\Pi}(\boldsymbol{\ell}) = \boldsymbol{\ell}$, $\boldsymbol{\Pi}(\boldsymbol{n}) = 0$, and $\boldsymbol{\Pi}^*(\underline{\boldsymbol{\ell}}) = 0$, $\boldsymbol{\Pi}^*(\underline{\boldsymbol{n}}) = \underline{\boldsymbol{n}}$.

1.2.3 Geodesic Kinematics

The characteristic feature of null hypersurfaces comes through understanding the kinematics through the derivatives of the null normals, along geodesic trajectories. We first extend the definition of the Weingarten map in Eq. (1.2.3) of $\mathscr{H}$ along $\boldsymbol{\ell}$, to all vectors of $T_p\mathscr{M}$, by setting

$$\chi(\boldsymbol{v}) = \chi\big|_{\mathscr{H}}(\boldsymbol{\Pi}(\boldsymbol{v})) = \nabla_{\boldsymbol{\Pi}(\boldsymbol{v})}\boldsymbol{\ell}, \tag{1.2.11}$$

which projects the mapping onto $T_p\mathscr{H}$. For the (extended) Weingarten map, we have the eigenvector $\boldsymbol{\ell}$ with the associated eigenvalue $\kappa_{(\boldsymbol{\ell})}$, since $\chi(\boldsymbol{\ell}) = \kappa_{(\boldsymbol{\ell})}\boldsymbol{\ell}$ as earlier, and additionally, $\chi(\boldsymbol{n}) = 0$.

Similarly, the projector dual $\boldsymbol{\Pi}^*$ can help in extending the definition of the second fundamental form $\boldsymbol{\Theta}$, defined in Eq. (1.2.4), onto the generalized deformation tensor $\boldsymbol{\Theta}$, with respect to the normal $\boldsymbol{\ell}$,

$$\boldsymbol{\Theta} = \boldsymbol{\Pi}^*\left(\boldsymbol{\Theta}\big|_{\mathscr{H}}\right), \tag{1.2.12a}$$

which acts on vectors in $T_p\mathscr{M}$ and projects onto the submanifold $T_p\mathscr{N}$. This can be expanded by noting Eq. (1.2.4b), and utilizing the relation in Eq. (1.2.10), $\boldsymbol{\Pi}(\boldsymbol{v}) = \vec{\boldsymbol{h}}(\boldsymbol{v}) - (\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{\ell}$, such that we have,

$$\boldsymbol{\Theta}(\boldsymbol{u},\boldsymbol{v}) = \vec{\boldsymbol{h}}(\boldsymbol{v})\cdot\nabla_{\vec{\boldsymbol{h}}(\boldsymbol{v})}\boldsymbol{\ell} = \nabla\underline{\boldsymbol{\ell}}(\vec{\boldsymbol{q}}(\boldsymbol{u}),\vec{\boldsymbol{q}}(\boldsymbol{v})), \tag{1.2.12b}$$

which can be understood as the projection of the bilinear form $\nabla\underline{\boldsymbol{\ell}}$ onto the cross section, given by

$$\boldsymbol{\Theta} = \vec{\boldsymbol{q}}^{*}\nabla\underline{\boldsymbol{\ell}}. \tag{1.2.12c}$$

In index notation, this simplifies to $\Theta_{\alpha\beta} = h^{\mu}_{\alpha}h^{\nu}_{\beta}\nabla_{\mu}\ell_{\nu}$, realising its projection onto the cross section. The bilinear form $\boldsymbol{\Theta}$ is degenerate along the null normals, with $\boldsymbol{\Theta}(\boldsymbol{\ell},\boldsymbol{v}) = \boldsymbol{\Theta}(\boldsymbol{v},\boldsymbol{k}) = 0$. Further, for any vector in the plane orthogonal to $\mathscr{N}$, the bilinear form is null.

To express the spacetime covariant derivative of the null normal, we begin by expanding the deformation tensor using Eq. (1.2.9a) as

$$\begin{aligned}\boldsymbol{\Theta}(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{\Pi}(\boldsymbol{u})\cdot\nabla_{\boldsymbol{\Pi}(\boldsymbol{v})}\boldsymbol{\ell} = (\boldsymbol{u}+(\boldsymbol{\ell}\cdot\boldsymbol{u})\boldsymbol{n})\cdot\boldsymbol{\chi}(\boldsymbol{v})\\ &= \boldsymbol{u}\cdot\nabla_{\boldsymbol{v}}\underline{\boldsymbol{\ell}} + (\boldsymbol{\ell}\cdot\boldsymbol{v})\boldsymbol{u}\cdot\nabla_{\boldsymbol{n}}\boldsymbol{\ell}\\ &= \nabla\underline{\boldsymbol{\ell}}(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{\ell}\cdot\boldsymbol{v})(\nabla_{\boldsymbol{n}}\cdot\boldsymbol{u}) + (\boldsymbol{\ell}\cdot\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{\chi}(\boldsymbol{v}),\end{aligned} \tag{1.2.13a}$$

from which we can obtain the general expression,

$$\nabla\underline{\boldsymbol{\ell}} = \boldsymbol{\Theta} + \underline{\boldsymbol{\ell}}\otimes\boldsymbol{\omega} - \nabla_{\boldsymbol{n}}\underline{\boldsymbol{\ell}}\otimes\underline{\boldsymbol{\ell}}, \tag{1.2.13b}$$

which in indexed notation is $\nabla_{\mu}\ell_{\nu} = \Theta_{\mu\nu} + \omega_{\mu}\ell_{\nu} - \ell_{\mu}n^{\alpha}\nabla_{\alpha}\ell_{\nu}$, where we have defined the *rotation 1-form* $\boldsymbol{\omega}$ defined by

$$\boldsymbol{\omega}(\boldsymbol{v}) = -\boldsymbol{n}\cdot\boldsymbol{\chi}(\boldsymbol{v}) = -\boldsymbol{n}\cdot\nabla_{\boldsymbol{\Pi}(\boldsymbol{v})}\boldsymbol{\ell}. \tag{1.2.14a}$$

Since the projector $\boldsymbol{\Pi}$ is involved in its definition, the 1-form $\boldsymbol{\omega}$ does not depend on the extension of the vector field $\boldsymbol{\ell}$ away from $\mathscr{H}$, similar to the deformation tensor $\boldsymbol{\Theta}$. Further, from the action on a tangent vector, we can deduce the functional expression for the rotation 1-form as

$$\boldsymbol{\omega} = -\boldsymbol{n}\cdot\nabla\underline{\boldsymbol{\ell}} - (\boldsymbol{n}\cdot\nabla_{\boldsymbol{n}}\boldsymbol{\ell})\underline{\boldsymbol{\ell}}. \tag{1.2.14b}$$

We also note the relevant relations for the action of the rotation 1-form on the null normal $\boldsymbol{\ell}$, as $\boldsymbol{\omega}(\boldsymbol{\ell}) = \kappa_{(\boldsymbol{\ell})}$ which reduces to the geodesic equation, and the degeneracy in the other null normal $\boldsymbol{n}$, leading to $\boldsymbol{\omega}(\boldsymbol{n}) = 0$. Also, we note the effect of rescaling the null normal, $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell}$, leads to $\boldsymbol{\omega}' = \boldsymbol{\omega} + \boldsymbol{\Pi}^{*}d\ln\alpha$, where $\boldsymbol{\Pi}$ remains invariant.

Further, we proceed to generalise the notion of physical variations along hypersurfaces through the deformation rate $\boldsymbol{\Theta}$ of $\mathscr{N}$ along $\boldsymbol{\ell}$, by splitting the deformation rate onto its fundamental components,

$$\boldsymbol{\Theta} = \frac{1}{n-2}\theta\boldsymbol{h} + \boldsymbol{\sigma} + \boldsymbol{\omega}, \tag{1.2.15a}$$

where we can define the expansion $\theta_{(\boldsymbol{\ell})}$, as the reduced trace in the cross section, given by,

$$\theta_{(\boldsymbol{\ell})} = \mathrm{Tr}\vec{\boldsymbol{\Theta}} = \Theta^{\mu}_{\mu} \tag{1.2.15b}$$

$$= g^{\mu\nu}\Theta_{\mu\nu} = h^{\mu\nu}\Theta_{\mu\nu} = h^{\mu\nu}\nabla_{\mu}\ell_{\nu} \tag{1.2.15c}$$

$$= \nabla\cdot\boldsymbol{\ell} - \kappa_{(\boldsymbol{\ell})}, \tag{1.2.15d}$$

further relating the divergence of $\boldsymbol{\ell}$. Also, the trace-free symmetric part is the *shear tensor* of $\mathscr{N}$ along $\boldsymbol{\ell}$, is given by,

$$\sigma_{\alpha\beta} = \Theta_{(\alpha\beta)} - \frac{1}{n-2}\theta_{(\boldsymbol{\ell})}h_{\alpha\beta}, \tag{1.2.15e}$$

and the antisymmetric part defines the *twist tensor*, as,

$$\tilde{\omega}_{\alpha\beta} = \Theta_{[\alpha\beta]}. \tag{1.2.15f}$$

Through scaling arguments, we can realise that the deformation rate $\boldsymbol{\Theta}$ and the shear tensor $\boldsymbol{\sigma}$ depend on the choice of the null normal $\boldsymbol{\ell}$ to $\mathscr{H}$. Further, the above tensors depend on the choice of the cross section $\mathscr{N}$ to $\mathscr{H}$, unlike the expansion $\theta_{(\ell)}$, which is independent of the choice of $\mathscr{N}$.

1.2.4 Dynamics of Null hypersurfaces

We proceed to relate the evolution equation for the change in the expansion $\theta_{(\boldsymbol{\ell})}$ along the null generators of $\mathscr{H}$, with the fundamental quantities of the deformation rate $\boldsymbol{\Theta}$. Any normal vector field $\boldsymbol{\ell}$ to a null hypersurface $\mathscr{H}$ of a n-dimensional spacetime $(\mathscr{M}, \boldsymbol{g})$, obeys the *Raychaudhuri equation*, given by

$$\nabla_{\boldsymbol{\ell}}\theta_{(\boldsymbol{\ell})} = \kappa_{(\boldsymbol{\ell})}\theta_{(\boldsymbol{\ell})} - \frac{1}{n-2}\theta^2_{(\boldsymbol{\ell})} - \sigma_{\mu\nu}\sigma^{\mu\nu} - \boldsymbol{R}(\boldsymbol{\ell},\boldsymbol{\ell}). \tag{1.2.16}$$

This characterises the evolution of the expansion parameter along the null generators of $\mathscr{H}$, given $\boldsymbol{\ell}$ is future-directed. Note that the above Eq. (1.2.16) is a particular case of the Raychaudhuri equation, where the twist $\tilde{\omega}$ defined by the antisymmetric component of the deformation rate $\boldsymbol{\Theta}$ of the vector field $\boldsymbol{\ell}$, vanishes. This appears because $\boldsymbol{\ell}$ is hypersurface-orthogonal, being normal to $\mathscr{H}$, and follows from *Frobenius theorem* in differential geometry. The Raychaudhuri equation, containing contracted products, is independent of the choice of the cross section.

If we choose the null normal $\boldsymbol{\ell}$ associated to an affine parameterisation of the null generators of $\mathscr{H}$, then $\kappa_{(\boldsymbol{\ell})} = 0$, and the null Raychaudhuri equation (1.2.16) simplifies further considering the positivity of the terms. By the *null energy condition* described on the total energy-momentum tensor in general relativity, we can further realise that

$$\nabla_{\boldsymbol{\ell}}\theta_{(\boldsymbol{\ell})} + \frac{1}{n-2}\theta^2_{(\boldsymbol{\ell})} \leq 0, \tag{1.2.17}$$

implying that the expansion $\theta_{(\boldsymbol{\ell})}$ along any null normal $\boldsymbol{\ell}$ associated to an affine parameterization of $\mathscr{H}$'s null generators cannot increase along $\boldsymbol{\ell}$.

1.3 Initial Value Formulation

General Relativity treats spacetime as a unified entity, whereas most of physics considers space and time separately. The equations of motion typically predict the future behaviour of a system based on its initial state. In physical theories, this predictive capability is often expressed by reformulating the dynamical equations into an initial-value form. This means that if we know the system's initial state, which satisfies specific initial conditions at a given moment in time, along with the evolution equations for the system, we can foresee the system's state at a later moment. This notion of time evolution is not naturally embedded in the theory, but there exist some methods to specify initial data as a Cauchy problem to Einstein's field equations, by slicing spacetime into spatial and temporal components.

1.3.1 Cauchy Problem on Spacelike Hypersurfaces

We do not have a global notion of time coordinate in general relativity due to diffeomorphisms between space and time coordinates. To introduce the notion of an initial value formulation, we must impose the foliation of the spacetime. We begin with the notion of a *Cauchy slice* Σ, as a spacelike hypersurface in a spacetime $\Sigma \subset \mathscr{M}$, such that every inextendible causal curve in $\mathscr{M}$ intersects Σ exactly once. A spacetime $(\mathscr{M}, \boldsymbol{g})$ admitting a Cauchy slice Σ is termed *globally hyperbolic*, and has topology $\Sigma \times \mathbb{R}$, which induces the notion of time.

Building upon, for a globally hyperbolic spacetime, we have a family of spacelike hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$, such that we define a *foliation* as a family of non-intersecting $\Sigma_t \cap \Sigma_{t'} = \emptyset$ for $t \neq t'$ spacelike hypersurfaces, with the foliation covering the entire spacetime, as $\mathscr{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$.

Considering Σ_t as the level sets of some smooth scalar field t, such that the gradient dt is timelike, we denote $\boldsymbol{n}$ as the future-directed timelike unit vector normal to Σ_t, such that the dual 1-form $\underline{\boldsymbol{n}}$, is parallel to the gradient field,

$$\underline{\boldsymbol{n}} = -N\, dt, \tag{1.3.1}$$

where the proportionality factor N is the *lapse function*, which ensures the normalization, $\boldsymbol{n} \cdot \boldsymbol{n} = -1$.

On $\mathscr{M}$, we can further introduce a 3+1 coordinate system adapted to Σ_t foliation by considering, on each hypersurface Σ_t, a coordinate system $\{x^i\}$, such that x^i varies smoothly from one surface to another. We thereby have a well-behaved coordinate system (t, x^i) on $\mathscr{M}$, where the coordinate time vector $\partial_t = \frac{\partial}{\partial t}$ is the effective dual to the gradient 1-form dt, as in $\boldsymbol{g}(dt, \partial_t) = 1$. Since we note $\boldsymbol{n} \cdot \partial_t = -1$ from the normalization condition, we use Eq. (1.3.1) to arrive at the orthogonal 3+1 decomposition,

$$\partial_t = N\boldsymbol{n} + \boldsymbol{N}, \tag{1.3.2}$$

where $\boldsymbol{N}$ is the *shift vector* of the coordinate system satisfying, $\boldsymbol{n} \cdot \boldsymbol{N} = 0$.

Given this, we have the components $g_{\mu\nu}$ of the metric tensor $\boldsymbol{g}$, expressed in the coordinates (t, x^i) in terms of the lapse N and the components N^i of the shift vector $\boldsymbol{N}$, through the line element, as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + q_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{1.3.3}$$

where q_{ij} are the components of the spatial metric $\boldsymbol{q}$, defined through the first fundamental form, as the induced metric,

$$\boldsymbol{q} = \boldsymbol{g} + \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}, \tag{1.3.4}$$

which is positive definite. Further, we can relate $g = \det(\boldsymbol{g})$ and $q = \det(\boldsymbol{q})$, through the relation, $\sqrt{-g} = N\sqrt{q}$.

The metric naturally induces an endomorphism on $T_p\mathscr{M}$ canonically associated with the metric, as the orthogonal projector onto Σ_t, given by,

$$\boldsymbol{\Pi}(\boldsymbol{v}) = \boldsymbol{v} + (\boldsymbol{n} \cdot \boldsymbol{v})\,\boldsymbol{n}, \tag{1.3.5}$$

whose existence is uniquely defined, unlike for null hypersurfaces, which can project quantities on the hypersurface.

Subsequently, there exists a unique torsion-free connection $\boldsymbol{D}$, on Σ_t associated with the metric $\boldsymbol{q}$, as $\boldsymbol{D}\boldsymbol{q} = 0$, which amounts to the pull-back of the covariant derivative using the projector dual $\boldsymbol{\Pi}^*$ as

$$\boldsymbol{D}\boldsymbol{v} = \boldsymbol{\Pi}^*\left(\nabla\boldsymbol{v}\right), \tag{1.3.6}$$

for a generic vector field $\boldsymbol{v}$, termed as the *Levi-Civita connection* associated with the metric $\boldsymbol{q}$.

As previously, we can define the Weingarten map as in Eq. (1.2.3), but with respect to the null normal $\boldsymbol{n}$, as the endomorphism on $T_p\Sigma_t$, as

$$\boldsymbol{\mathcal{K}}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}}\boldsymbol{n}, \tag{1.3.7}$$

which can be shown to be well-defined and self-adjoint with respect to the metric $\boldsymbol{q}$. We can also, orthogonally extend the map to $T_p\mathscr{M}$, with the orthogonal projector $\boldsymbol{\Pi}$ in Eq. (1.3.5), by setting $\boldsymbol{\mathcal{K}}(\boldsymbol{v}) = \nabla_{\boldsymbol{\Pi}(\boldsymbol{v})}\boldsymbol{n}$.

Further, we have the *extrinsic curvature tensor* $\boldsymbol{K}$ of the hypersurface, as the (negative) second fundamental form, given by

$$\boldsymbol{K}(\boldsymbol{u}, \boldsymbol{v}) = -\boldsymbol{u} \cdot \boldsymbol{\mathcal{K}}(\boldsymbol{v}), \tag{1.3.8a}$$

which can be further simplified, due to the self-adjointness leading to a symmetry, as in Eq. (1.2.4), to arrive at

$$\boldsymbol{K} = -\boldsymbol{\Pi}^*\left(\nabla\underline{\boldsymbol{n}}\right), \tag{1.3.8b}$$

where we can substitute the explicit expression of the metric in Eq. (1.3.4) to arrive at

$$\boldsymbol{K} = -\nabla \underline{\boldsymbol{n}} - \boldsymbol{D} \ln N \otimes \underline{\boldsymbol{n}}, \tag{1.3.8c}$$

where we realise that $\nabla_{\boldsymbol{n}} \underline{\boldsymbol{n}} = \boldsymbol{D} \ln N$ since $\boldsymbol{\mathcal{K}}(\boldsymbol{n}) = 0$.

Now, we proceed with deriving local quantities on Σ_t, through projections arriving at the *Gauss-Codazzi relations*, as

$$\Pi^{\alpha}_{\mu}\Pi^{\nu}_{\beta}\Pi^{\rho}_{\gamma}\Pi^{\sigma}_{\delta}R^{\mu}_{\nu\rho\sigma} = {}^{(3)}R^{\alpha}_{\beta\gamma\delta} + K^{\alpha}_{\gamma}K_{\beta\delta} - K^{\alpha}_{\delta}K_{\beta\gamma}, \tag{1.3.9a}$$

$$\Pi^{\alpha}_{\mu}\Pi^{\nu}_{\beta}\Pi^{\rho}_{\gamma}n^{\sigma}R^{\mu}_{\nu\rho\sigma} = D_{\beta}K^{\alpha}_{\gamma} - D^{\alpha}K_{\beta\gamma}. \tag{1.3.9b}$$

The above relations are derived from projecting the intrinsic Riemann tensor ${}^{(3)}\boldsymbol{\mathcal{R}}$, and using the commutation of two covariant derivatives with respect to the connection $\boldsymbol{D}$, analogous to Eq. (1.1.3), as,

$$ {}^{(3)}\boldsymbol{\mathcal{R}}(\boldsymbol{\omega}, \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{q}\left(\boldsymbol{\omega}, \boldsymbol{D_u D_v w} - \boldsymbol{D_v D_u w} - \boldsymbol{D}_{[\boldsymbol{u},\boldsymbol{v}]}\boldsymbol{w}\right), \tag{1.3.10a}$$

which can be explicitly defined through its components, ${}^{(3)}R_{\alpha\beta\gamma\delta}$, given by the projected Ricci identity as,

$$\left(D_{\alpha}D_{\beta} - D_{\beta}D_{\alpha}\right)w^{\gamma} = {}^{(3)}R^{\gamma}_{\mu\alpha\beta}w^{\mu}. \tag{1.3.10b}$$

Further, if we contract the Gauss relation above, and use the idempotence of the metric projections, and further take its trace with respect to $\boldsymbol{q}$, we obtain

$$R + 2R_{\mu\nu}n^{\mu}n^{\nu} = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}, \tag{1.3.11}$$

where $K = q^{\mu\nu}K_{\mu\nu}$. This fundamental relation connects the intrinsic curvature of Σ_t, represented by the Ricci scalar ${}^{(3)}R$, to its extrinsic curvature, represented by $K^2 - K_{\mu\nu}K^{\mu\nu}$.

We now project the Einstein's equation in Eq. (1.1.7), using the projector operator $\boldsymbol{\Pi}^*$, to arrive at the dynamical equation, given as

$$\mathcal{L}_{N\boldsymbol{n}}K_{\alpha\beta} = -D_{\alpha}D_{\beta}N + N\left[{}^{(3)}R_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}_{\beta} + KK_{\alpha\beta} + 4\pi((S-E)q_{\alpha\beta} - 2S_{\alpha\beta}\right] \tag{1.3.12}$$

where the energy density, $E = \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n})$, that is $E = T_{\mu\nu}n^{\mu}n^{\nu}$, and $\boldsymbol{S} = \boldsymbol{\Pi}^*(\boldsymbol{T})$ such that $S_{\alpha\beta} = \Pi^{\mu}_{\alpha}\Pi^{\nu}_{\beta}S_{\mu\nu}$ is the strain tensor with $S = h^{\alpha\beta}S_{\alpha\beta}$ being its trace, which are projected components of the matter stress-energy tensor. We can also express the extrinsic curvature in Eq. (1.3.8) in terms of the Lie derivative of the metric $\boldsymbol{q}$ as its evolution equation given by,

$$\mathcal{L}_{N\boldsymbol{n}}q_{\mu\nu} = -2NK_{\mu\nu}. \tag{1.3.13}$$

Further, the orthogonal projections of the Einstein equation lead to *constraint* equations, which are given by

$${}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi E, \tag{1.3.14a}$$

$$D^{\mu}K_{\alpha\mu} - D_{\alpha}K = 8\pi J_{\alpha}, \tag{1.3.14b}$$

where E corresponds to *Hamiltonian constraint*, and the *momentum constraint* is characterized by $\boldsymbol{J}(\boldsymbol{v}) = -\boldsymbol{T}\,(\boldsymbol{n}, \boldsymbol{\Pi}(\boldsymbol{v}))$, equivalently, $J_{\alpha} = -\Pi^{\mu}_{\alpha} T_{\mu\nu} n^{\nu}$, which is the momentum density.

Since the constraint equations in Eq. (1.3.14), do not contain any second-order derivatives of the metric in a timelike direction, unlike the dynamical equation in Eq. (1.3.12), they are not associated with the dynamical evolution of the gravitational field, and represent constraints to be satisfied by the metric $\boldsymbol{q}$, and the extrinsic curvature $\boldsymbol{K}$, on every hypersurface Σ_t.

The initial data problem consists of specifying the values of $\boldsymbol{q}$ and $\boldsymbol{K}$ on some initial spatial Cauchy hypersurface Σ_0, and then evolving them according to Eq. (1.3.12) and Eq. (1.3.13). At each point, including the initial data, we must satisfy the constraint relations in Eq. (1.3.14), and such a well-posed initial value formulation exists.

By using coordinates adapted to the foliation, we can transform the system of tenosrial equations into partial differential equations, through the natural basis of $T_p\mathscr{M}$ as (∂_t, ∂_i) as the directional derivatives. The time vector ∂_t is tangent to the lines of constant spatial coordinates, and the vector ∂_i is tangent to lines in the hypersurface, such that it belongs in $T_p\Sigma_t$. With this, the Einstein equation reduces to a second-order non-linear partial differential equation system for the unknowns $\{\boldsymbol{q}, \boldsymbol{K}, N, \boldsymbol{N}\}$. Using this framework, the Hamiltonian and momentum constraints become a system of elliptic partial differential equations on each submanifold Σ_t, whereas the evolution equations take the form of hyperbolic partial differential equations.

Given a set $\{\Sigma_0, \boldsymbol{q}, \boldsymbol{K}, E, \boldsymbol{p}\}$, where Σ_0 is a 3-dimensional manifold, $\boldsymbol{q}$ is the Riemannian metric on Σ_0, $\boldsymbol{K}$ is a symmetric bilinear form on Σ_0, E and $\boldsymbol{p}$ respectively scalar and vector fields on Σ_0, which obeys the constraint equations in Eq. (1.3.14), there exists a spacetime $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{T})$ satisfying the Einstein equation, and Σ_0 can be embedded as an hypersurface of $\mathscr{M}$, with induced metric $\boldsymbol{q}$ and extrinsic curvature $\boldsymbol{K}$. This is the formulation of the Cauchy initial value problem on spacelike hypersurfaces in Relativity.

1.3.2 Characteristic Problem on Null Hypersurfaces

Unlike the Cauchy problem described earlier on spacelike hypersurfaces, the Characteristic problem deals with initial data on one or more null hypersurfaces, which can be written as ordinary differential equations, which we refer to as propagation equations, with simpler constraints. The Cauchy problem is natural for problems relating to the space-like infinity, whereas the Characteristic problem is suited for questions concerning black hole horizon neighbourhoods and null infinity with the study of gravitational radiation.

We now consider two smooth null hypersurfaces $\mathscr{H}_1$ and $\mathscr{H}_2$, which intersect transversely in a codimension-2 submanifold $\mathscr{N}$ which is spacelike.

The characteristic initial value problem involves the choice of data on $\mathscr{H}_1 \cup \mathscr{H}_2$, such that the solution can be determined, and the determination of the spacetime to the future of $\mathscr{H}_1 \cup \mathscr{H}_2$. Near the hypersurfaces, we can choose adapted coordinates $\{x^1, x^2, x^\alpha\}$, such that N_i corresponds to the set $\{x_i : x_i = 0\}$ for $i = 1, 2$. Under some conditions for the metric on the hypersurface to be characteristic, it can be shown that the Raychaudhuri equation in Eq. (1.2.16) provides a natural constraint equation for the null generators of the hypersurfaces.

The set of constraint equations forms a hierarchical system of ordinary differential equations along the null generators. This system is solved sequentially, or propagated from the intersection surface $\mathscr{N}$ outwards. In the formulation we work with, we specify the conformal class of the metric on the cross-section $\mathscr{N}$, as the initial data. This is represented by a fiducial metric $\tilde{\boldsymbol{q}}$, which relates to the induced 2-metric $\boldsymbol{q}$ by the relation,

$$\boldsymbol{q} = \Omega^2(x^1, x^2)\tilde{\boldsymbol{q}}, \tag{1.3.15}$$

where $\Omega(x^1, x^2)$ is the conformal factor as a non-negative scalar field on $\mathscr{M}$. The power of the conformal method lies in its transformation of the primary constraint equation. By substituting the ansatz metric and the requisite expansion in Eq. (1.2.15d) and shear in Eq. (1.2.15e) into the non-linear Raychaudhuri equation in Eq. (1.2.16), we arrive at a second-order ordinary differential equation for the conformal factor Ω. We can integrate the equation along null generators of the hypersurfaces $\mathscr{H}_1$ and $\mathscr{H}_2$ given initial data on $\mathscr{N}$.

Chapter 2

Spinorial Coordinates

2.1 Tetrad Formalism

At each point of spacetime, we set up a basis of four contravariant vectors,

$$e^{\mu}_{(a)} \quad (a = 1, 2, 3, 4), \tag{2.1.1}$$

where (a) denotes the tetrad indices. The choice of the tetrad basis is characterized by the underlying symmetries of spacetime, and we project relevant quantities onto the chosen basis.

Associated with the contravariant vectors in (2.1.1), we have the covariant vectors

$$e_{(a)\mu} = g_{\mu\nu} e^{\nu}_{(a)}, \tag{2.1.2}$$

lowered by the metric tensor $g_{\mu\nu}$.

In addition, we define the analogous tetrad inverse $e^{(b)}_{\mu}$ of the matrix $[e^{\mu}_{(a)}]$ with the tetrad index labelling the rows and the tensor index labelling the columns, such that

$$e^{\mu}_{(a)} e^{(b)}_{\mu} = \delta^{(b)}_{(a)}, \tag{2.1.3a}$$

$$e^{\mu}_{(a)} e^{(a)}_{\nu} = \delta^{\mu}_{\nu}. \tag{2.1.3b}$$

Note that we obtain the relation, relating the spacetime metric that completely defines the inverse,

$$e_{(a)\mu} e^{(a)}_{\nu} = g_{\mu\gamma} e^{\gamma}_{(a)} e^{(a)}_{\nu} = g_{\mu\nu}. \tag{2.1.4}$$

Further, as part of the definitions, we define a constant symmetric matrix, $\eta_{(a)(b)}$,

$$e^{\mu}_{(a)} e_{(b)\mu} = \eta_{(a)(b)}, \tag{2.1.5}$$

If we suppose the basis vectors $e^{\mu}_{(a)}$ are orthonormal, we would have $\eta_{(a)(b)} = \mathrm{diag}(+1, -1, -1, -1)$.

Defining the inverse $\eta^{(a)(b)}$ of the matrix $[\eta_{(a)(b)}]$ from Eq. (2.1.5), we have

$$\eta^{(a)(b)} \eta_{(b)(c)} = e^{\mu(a)} e^{(b)}_{\mu} e^{\nu}_{(b)} e_{(c)\nu}$$

$$= e^{\mu(a)} \delta^{\nu}_{\mu} e_{(c)\nu} = \delta^{(a)}_{(c)}. \tag{2.1.6}$$

Further, we have the raising and lowering of tetrad indices,

$$\eta_{(a)(b)} e^{(a)}_{\mu} = e^{\nu}_{(a)} e_{(b)\nu} e^{(a)}_{\mu} = e_{(b)\nu} e^{\nu}_{(a)} e^{(a)}_{\mu} = e_{(b)\mu}, \tag{2.1.7a}$$

$$\eta^{(a)(b)} e_{(a)\mu} = e^{(a)\nu} e^{(b)}_{\nu} e_{(a)\mu} = e^{(b)}_{\nu} e^{(a)\nu} e_{(a)\mu} = e^{(b)}_{\mu}. \tag{2.1.7b}$$

Now, given any vector field, we obtain the tetrad components by projecting it into the chosen tetrad frame as

$$A_{(a)} = e_{(a)\mu} A^{\mu} = e^{\mu}_{(a)} A_{\mu} \tag{2.1.8a}$$

$$A^{(a)} = \eta^{(a)(b)} A_{(b)} = e^{(a)}_{\mu} A^{\mu} = e^{(a)\mu} A_{\mu}, \tag{2.1.8b}$$

with the vector components related as

$$A^{\mu} = e^{\mu}_{(a)} A_{(a)} = e^{(a)\mu} A_{(a)}. \tag{2.1.9}$$

For a general tensor field, we have

$$T_{(a)(b)} = e^{\mu}_{(a)} e^{\nu}_{(b)} T_{\mu\nu} = e^{\mu}_{(a)} T_{\mu(b)} \tag{2.1.10a}$$

$$T_{\mu\nu} = e^{(a)}_{\mu} e^{(b)}_{\nu} T_{(a)(b)} = e^{(a)}_{\mu} T_{(a)\nu}. \tag{2.1.10b}$$

From Eqs (2.1.3), (2.1.5), (2.1.6), (2.1.7), we realise that we can freely interchange tensor indices to the tetrad indices and vice versa; raise (lower) tetrad indices with $\eta^{(a)(b)}$ (with $\eta_{(a)(b)}$).

2.1.1 Directional Derivatives

The natural notion of tangent vectors is to analyze them as *directional derivatives* along integral curves. Analogously, we can define the directional derivatives considering the contravariant vectors as

$$\mathbf{e}_{(a)} = e^{\mu}_{(a)} \frac{\partial}{\partial x^{\mu}}. \tag{2.1.11}$$

such that, for a scalar field φ, we have

$$\partial_{(a)} \varphi = e^{\mu}_{(a)} \frac{\partial \varphi}{\partial x^{\mu}} = e^{\mu}_{(a)} \partial_{\mu} \varphi, \tag{2.1.12}$$

and in general

$$\partial_{(b)} A_{(a)} = e^{\mu}_{(b)} \partial_{\mu} A_{(a)}$$

$$= e^{\mu}_{(b)}\partial_{\mu}[e^{\nu}_{(a)}A_{\nu}] = e^{\mu}_{(b)}\nabla_{\mu}[e^{\nu}_{(a)}A_{\nu}]$$
$$= e^{\mu}_{(b)}\nabla_{\mu}A_{\nu}e^{\nu}_{(a)} + \nabla_{\mu}e_{(a)\nu}e^{\mu}_{(b)}e^{\nu}_{(c)}A^{(c)} \tag{2.1.13}$$

As a beautiful geometric insight of the directional derivative of the basis vectors along the tetrad directions projected onto the tetrad, we note the quantity

$$\nabla_{\mu}e_{(a)\nu} = e^{(c)}_{\nu}\gamma_{(c)(a)(b)}e^{(b)}_{\mu} \tag{2.1.14}$$

as the *Ricci rotation coefficients*, such that we can rewrite the directional derivative as

$$\partial_{(b)}A_{(a)} = e^{\mu}_{(b)}\nabla_{\mu}A_{\nu}e^{\nu}_{(a)} + \gamma_{(c)(a)(b)}A^{(c)}. \tag{2.1.15}$$

Since $\eta_{(a)(b)}$'s are constant in the tensorial coordinates, we have

$$\begin{aligned}
0 &= \partial_{\mu}\eta_{(a)(b)} = \nabla_{\mu}\eta_{(a)(b)} = \nabla_{\mu}[e^{\nu}_{(a)}e_{(b)\nu}] \\
&= (\nabla_{\mu}e^{\nu}_{(a)})e_{(b)\nu} + e^{\nu}_{(a)}(\nabla_{\mu}e_{(b)\nu}) \\
&= (e^{(c)\nu}\gamma_{(c)(a)(b)}e^{(b)}_{\mu})e_{(b)\nu} + e^{\nu}_{(a)}(e^{(c)}_{\nu}\gamma_{(c)(b)(a)}e^{(a)}_{\mu}) \\
&= e^{(c)\nu}\gamma_{(c)(a)(b)}\delta_{\mu\nu} + \delta^{(c)}_{(a)}\gamma_{(a)(c)(b)}e^{(a)}_{\mu} \\
&= e^{(c)}_{\mu}[\gamma_{(c)(a)(b)} + \gamma_{(a)(c)(b)}],
\end{aligned} \tag{2.1.16}$$

thus resulting in antisymmetry in the first pair of indices as,

$$\gamma_{(c)(a)(b)} = -\gamma_{(a)(c)(b)}. \tag{2.1.17}$$

This antisymmetry profoundly manifests the power of the tetrad formalism in representing the curvature through $\frac{n^2(n-1)}{2}$ independent components of the Ricci rotation coefficients in n-dimensional spacetime. Specifically, the relevance of 24 independent components in 4-dimensional spacetime is beneficial as compared to the 40 independent components of the Christoffel symbols.

Thereby, we can rewrite Eq. (2.1.14) as

$$\begin{aligned}
\nabla_{\mu}e^{\nu}_{(a)} &= e^{(c)\nu}\gamma_{(c)(a)(b)}e^{(b)}_{\mu} = -e^{(c)\nu}\gamma_{(a)(c)(b)}e^{(b)}_{\mu} \\
&= -e^{(c)\nu}\gamma_{(a)(c)\mu} = -\gamma^{\nu}_{(a)\mu}.
\end{aligned} \tag{2.1.18}$$

To understand the local nature of the connection coefficients $\gamma_{(a)(b)(c)}$, we note that the derivative difference can be manifested as the covariant derivative difference.

Consider the following term $\lambda_{(a)(b)(c)}$ defined as

$$\begin{aligned}
\lambda_{(a)(b)(c)} &= \partial_{\nu}e_{(b)\mu}e^{[\mu}_{(a)}e^{\nu]}_{(c)} \\
&= \partial_{[\nu}e_{(b)\mu]}e^{\mu}_{(a)}e^{\nu}_{(c)} = \nabla_{[\nu}e_{(b)\mu]}e^{\mu}_{(a)}e^{\nu}_{(c)}
\end{aligned}$$

$$= \gamma_{(a)(b)(c)} - \gamma_{(c)(b)(a)}. \tag{2.1.19}$$

where we have replaced with the covariant derivative for symmetric connections.

Thereby, from Eq. (2.1.19), we have the following relations

$$\lambda_{(a)(b)(c)} = \gamma_{(a)(b)(c)} - \gamma_{(c)(b)(a)} \tag{2.1.20a}$$

$$\lambda_{(c)(a)(b)} = \gamma_{(c)(a)(b)} - \gamma_{(b)(a)(c)} \tag{2.1.20b}$$

$$\lambda_{(b)(c)(a)} = \gamma_{(b)(c)(a)} - \gamma_{(a)(c)(b)}, \tag{2.1.20c}$$

which can be rearranged to arrive at,

$$\gamma_{(a)(b)(c)} = \frac{1}{2}\Big[\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}\Big]. \tag{2.1.21}$$

Thus, we have a completely local formulation of the connection coefficients with respect to the ordinary derivatives.

2.1.2 Intrinsic Derivative

Further, to understand the notion of the effective covariant derivative in the tetrad coordinates, we can rewrite Eq. (2.1.15) as

$$\begin{aligned} e^{\mu}_{(b)} \nabla_{\mu} A_{\nu} e^{\nu}_{(a)} &= \partial_{(b)} A_{(a)} - \gamma_{(c)(a)(b)} A^{(c)} \\ &= \partial_{(b)} A_{(a)} - \eta^{(c)(d)} \gamma_{(c)(a)(b)} A_{(d)}, \end{aligned} \tag{2.1.22}$$

where we subsequently denote it as the *intrinsic derivative* of $A_{(a)}$ in the direction $e_{(b)}$, as

$$\nabla_{(b)} A_{(a)} = e^{\mu}_{(a)} \nabla_{\mu} A_{\nu} e^{\nu}_{(b)}, \tag{2.1.23a}$$

$$\nabla_{\mu} A_{\nu} = e^{(a)}_{\mu} \nabla_{(b)} A_{(a)} e^{(b)}_{\nu}. \tag{2.1.23b}$$

Thereby, we have the equivalent notion of the covariant derivative in terms of the intrinsic derivative relating to the directional derivative from Eq. (2.1.22) as

$$\nabla_{(b)} A_{(a)} = \partial_{(b)} A_{(a)} - \eta^{(c)(d)} \gamma_{(c)(a)(b)} A_{(d)}. \tag{2.1.24}$$

The intrinsic derivative of the Riemann tensor in the tetrad coordinates is given by

$$\begin{aligned} \nabla_{(f)} R_{(a)(b)(c)(d)} &= e^{\mu}_{(f)} \nabla_{\mu} R_{\nu(b)(c)(d)} e^{\nu}_{(a)} \\ &= e^{\mu}_{(f)} \nabla_{\mu} R_{\nu\gamma\alpha\beta} e^{\nu}_{(a)} e^{\gamma}_{(b)} e^{\alpha}_{(c)} e^{\beta}_{(d)}. \end{aligned} \tag{2.1.25}$$

2.1.3 Riemann Tensor Projection

By the definition of the Riemann tensor, we note

$$R_{\mu\nu\alpha\beta}e^{\mu}_{(a)} = \nabla_{\alpha}\nabla_{\beta}e_{(a)\nu} - \nabla_{\beta}\nabla_{\alpha}e_{(a)\beta}, \tag{2.1.26}$$

which, on projection with the tetrad frame, we have,

$$\begin{aligned}
R_{(a)(b)(c)(d)} &= R_{\mu\nu\alpha\beta}e^{\mu}_{(a)}e^{\nu}_{(b)}e^{\alpha}_{(c)}e^{\beta}_{(d)} \\
&= \nabla_{\alpha}\Big(\nabla_{\beta}e_{(a)\nu}\Big)e^{\nu}_{(b)}e^{\alpha}_{(c)}e^{\beta}_{(d)} - \nabla_{\beta}\Big(\nabla_{\alpha}e_{(a)\nu}\Big)e^{\nu}_{(b)}e^{\alpha}_{(c)}e^{\beta}_{(d)} \\
&= \nabla_{\alpha}\Big(e^{(f)}_{\nu}\gamma_{(f)(a)(g)}e^{(g)}_{\beta}\Big)e^{\nu}_{(b)}e^{\alpha}_{(c)}e^{\beta}_{(d)} - \nabla_{\beta}\Big(e^{(f)}_{\nu}\gamma_{(f)(a)(g)}e^{(g)}_{\alpha}\Big)e^{\nu}_{(b)}e^{\alpha}_{(c)}e^{\beta}_{(d)},
\end{aligned} \tag{2.1.27}$$

Expanding out, and using Eq. (2.1.9)), Eq. (2.1.14), and Eq. (2.1.3), we have

$$\begin{aligned}
R_{(a)(b)(c)(d)} &= \nabla_{\alpha}e^{(f)}_{\nu}\gamma_{(f)(a)(d)}e^{\nu}_{(b)}e^{\alpha}_{(c)} + \nabla_{\alpha}\gamma_{(b)(a)(d)}e^{\alpha}_{(c)} + \gamma_{(b)(a)(g)}\nabla_{\alpha}e^{(g)}_{\beta}e^{\alpha}_{(c)}e^{\beta}_{(d)} \\
&\quad - \nabla_{\beta}e^{(f)}_{\nu}\gamma_{(f)(a)(c)}e^{\nu}_{(b)}e^{\beta}_{(d)} - \nabla_{\beta}\gamma_{(b)(a)(c)}e^{\beta}_{(d)} - \gamma_{(b)(a)(g)}\nabla_{\beta}e^{(g)}_{\alpha}e^{\alpha}_{(c)}e^{\beta}_{(d)} \\
&= e^{(h)}_{\nu}\gamma^{(f)}_{(h)(i)}e^{(i)}_{\alpha}\gamma_{(f)(a)(d)}e^{\nu}_{(b)}e^{\alpha}_{(c)} + \partial_{\alpha}\gamma_{(b)(a)(d)}e^{\alpha}_{(c)} \\
&\quad + \gamma_{(b)(a)(g)}e^{(h)}_{\beta}\gamma^{(g)}_{(h)(i)}e^{(i)}_{\alpha}e^{\alpha}_{(c)}e^{\beta}_{(d)} - e^{(h)}_{\nu}\gamma^{(f)}_{(h)(i)}e^{(i)}_{\beta}\gamma_{(f)(a)(c)}e^{\nu}_{(b)}e^{\beta}_{(d)} \\
&\quad - \partial_{\beta}\gamma_{(b)(a)(c)}e^{\beta}_{(d)} - \gamma_{(b)(a)(g)}e^{(h)}_{\alpha}\gamma^{(g)}_{(h)(i)}e^{(i)}_{\beta}e^{\alpha}_{(c)}e^{\beta}_{(d)} \\
&= \gamma^{(f)}_{(b)(c)}\gamma_{(f)(a)(d)} + \partial_{(c)}\gamma_{(b)(a)(d)} + \gamma_{(b)(a)(g)}\gamma^{(g)}_{(d)(c)} \\
&\quad - \gamma^{(f)}_{(b)(d)}\gamma_{(f)(a)(c)} - \partial_{(d)}\gamma_{(b)(a)(c)} - \gamma_{(b)(a)(g)}\gamma^{(g)}_{(c)(d)}
\end{aligned} \tag{2.1.28}$$

Thereby, we have

$$\begin{aligned}
R_{(a)(b)(c)(d)} &= -\partial_{(d)}\gamma_{(a)(b)(c)} + \partial_{(c)}\gamma_{(a)(b)(d)} + \gamma_{(b)(a)(f)}\Big[\gamma^{(f)}_{(c)(d)} \\
&\quad - \gamma^{(f)}_{(d)(c)}\Big] + \gamma^{(f)}_{(b)(d)}\gamma_{(f)(a)(c)} - \gamma^{(f)}_{(b)(c)}\gamma_{(f)(a)(d)}
\end{aligned} \tag{2.1.29}$$

Because of the antisymmetry of the rotation coefficients in the first pair of their indices, we have 36 equations of the kind Eq. (2.1.29).

Further, the intrinsic derivative $\partial_{(f)}R_{(a)(b)(c)(d)}$, using Eq. (2.1.25) and Eq. (2.1.3), equals

$$\begin{aligned}
\partial_{(f)}R_{(a)(b)(c)(d)} &= e^{\mu}_{(f)}\nabla_{\mu}[R_{(a)(b)(c)(d)}] \\
&= \nabla_{(f)}R_{(a)(b)(c)(d)} + \eta^{(p)(q)}\Big[\gamma_{(p)(a)(f)}R_{(q)(b)(c)(d)} \\
&\quad + \gamma_{(p)(b)(f)}R_{(q)(b)(c)(d)} + \gamma_{(p)(a)(f)}R_{(q)(b)(c)(d)} \\
&\quad + \gamma_{(p)(a)(f)}R_{(q)(b)(c)(d)}\Big],
\end{aligned} \tag{2.1.30}$$

which we can rewrite as

$$\begin{aligned}\nabla_{(f)}R_{(a)(b)(c)(d)} = \partial_{(f)}R_{(a)(b)(c)(d)} \\ -\eta^{(p)(q)}\Big[\gamma_{(p)(a)(f)}R_{(q)(b)(c)(d)} + \gamma_{(p)(b)(f)}R_{(q)(b)(c)(d)} \\ + \gamma_{(p)(a)(f)}R_{(q)(b)(c)(d)}\gamma_{(p)(a)(f)}R_{(q)(b)(c)(d)}\Big]\end{aligned} \tag{2.1.31}$$

2.2 Newman-Penrose formalism

The Newman-Penrose tetrad formalism [1] is a special choice of basis vectors to manifest the spinorial symmetries of general relativity through a null basis. The null vectors defined earlier, $\boldsymbol{\ell}$ and $\boldsymbol{n}$ in Eq. (1.2.7), comprising $T_p^{\perp}\mathscr{N} = \mathrm{Span}(\boldsymbol{\ell},\boldsymbol{n})$ are extended with two orthonormal vectors in $T_p\mathscr{N}$, which are complex conjugated by superpositions. We have the basis vectors as $\{\boldsymbol{\ell},\boldsymbol{n},\boldsymbol{m},\bar{\boldsymbol{m}}\}$, where $\boldsymbol{\ell}$ and $\boldsymbol{n}$ are real null vectors, and $\boldsymbol{m},\bar{\boldsymbol{m}}$ are complex conjugates of each other. Because there is no complete real basis of null directions in a Lorentzian manifold, two of these vectors have to be complex, and since the final metric is real, these two complex tetrads are complex conjugate. The two real tetrads label particular ingoing and outgoing null directions, so this formalism is well adapted to describe geometrically the propagation of gravitational radiation.

2.2.1 Null Basis

We have the orthonormal conditions

$$\boldsymbol{\ell}\cdot\boldsymbol{m} = \boldsymbol{\ell}\cdot\bar{\boldsymbol{m}} = \boldsymbol{n}\cdot\boldsymbol{m} = \boldsymbol{n}\cdot\bar{\boldsymbol{m}} = 0, \tag{2.2.1}$$

and the null requirements,

$$\boldsymbol{\ell}\cdot\boldsymbol{\ell} = \boldsymbol{m}\cdot\boldsymbol{m} = \bar{\boldsymbol{m}}\cdot\bar{\boldsymbol{m}} = \boldsymbol{n}\cdot\boldsymbol{n} = 0. \tag{2.2.2}$$

We further introduce the following normalization conditions on the basis vectors as

$$\boldsymbol{\ell}\cdot\boldsymbol{n} = 1, \quad \boldsymbol{m}\cdot\bar{\boldsymbol{m}} = -1. \tag{2.2.3}$$

From Eq. (2.1.5), we have the fundamental representation $\eta_{(a)(b)}$ as a constant symmetric matrix of the form

$$[\eta_{(a)(b)}] = \begin{pmatrix} \boldsymbol{\ell}\cdot\boldsymbol{\ell} & \boldsymbol{\ell}\cdot\boldsymbol{n} & \boldsymbol{\ell}\cdot\boldsymbol{m} & \boldsymbol{\ell}\cdot\bar{\boldsymbol{m}} \\ \boldsymbol{n}\cdot\boldsymbol{\ell} & \boldsymbol{n}\cdot\boldsymbol{n} & \boldsymbol{n}\cdot\boldsymbol{m} & \boldsymbol{n}\cdot\bar{\boldsymbol{m}} \\ \boldsymbol{m}\cdot\boldsymbol{\ell} & \boldsymbol{m}\cdot\boldsymbol{n} & \boldsymbol{m}\cdot\boldsymbol{m} & \boldsymbol{m}\cdot\bar{\boldsymbol{m}} \\ \bar{\boldsymbol{m}}\cdot\boldsymbol{\ell} & \bar{\boldsymbol{m}}\cdot\boldsymbol{n} & \bar{\boldsymbol{m}}\cdot\boldsymbol{m} & \bar{\boldsymbol{m}}\cdot\bar{\boldsymbol{m}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.2.4}$$

further relating the inverse

$$[\eta_{(a)(b)}] = [\eta^{(a)(b)}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.2.5}$$

with the correspondence

$$\mathbf{e}_\ell = \boldsymbol{\ell}, \quad \mathbf{e}_n = \boldsymbol{n}, \quad \mathbf{e}_m = \boldsymbol{m}, \quad \text{and} \quad \mathbf{e}_{\bar{m}} = \bar{\boldsymbol{m}}, \tag{2.2.6}$$

with the corresponding covariant basis

$$\mathbf{e}^\ell = \eta^{\ell(a)}\mathbf{e}_{(a)} = -\boldsymbol{n}, \tag{2.2.7a}$$
$$\mathbf{e}^n = \eta^{n(a)}\mathbf{e}_{(a)} = -\boldsymbol{\ell}, \tag{2.2.7b}$$
$$\mathbf{e}^m = \eta^{m(a)}\mathbf{e}_{(a)} = \bar{\boldsymbol{m}}, \tag{2.2.7c}$$
$$\mathbf{e}^{\bar{m}} = \eta^{\bar{m}(a)}\mathbf{e}_{(a)} = \boldsymbol{m}. \tag{2.2.7d}$$

Thereby, the spacetime metric $\boldsymbol{g}$ can be written in terms of the complete basis $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$, as,

$$\boldsymbol{g} = -\boldsymbol{\ell} \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{\ell} + \boldsymbol{m} \otimes \bar{\boldsymbol{m}} + \bar{\boldsymbol{m}} \otimes \boldsymbol{m}, \tag{2.2.8}$$

and with index notation given by,

$$g_{\mu\nu} = \eta_{(a)(b)}\boldsymbol{e}^{(a)}_\mu \boldsymbol{e}^{(b)}_\nu = 2\Big(-\ell_{(\mu}n_{\nu)} + m_{(\mu}\bar{m}_{\nu)}\Big). \tag{2.2.9}$$

Further, since the tetrad $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ is complete, we have

$$\delta^\mu_\nu = -\ell^i n_\nu - n^\mu \ell_\nu + m^\mu \bar{m}_\nu + \bar{m}^\mu m_\nu. \tag{2.2.10}$$

The dual cotetrad can further be designated as the directional derivatives along the basis vectors with

$$\mathbf{e}_\ell = \ell^\mu \nabla_\mu = D, \tag{2.2.11a}$$
$$\mathbf{e}_n = n^\mu \nabla_\mu = \Delta, \tag{2.2.11b}$$
$$\mathbf{e}_m = m^\mu \nabla_\mu = \delta, \tag{2.2.11c}$$
$$\mathbf{e}_{\bar{m}} = \bar{m}^\mu \nabla_\mu = \bar{\delta}. \tag{2.2.11d}$$

We can project the intrinsic derivatives, such that the Ricci rotation coefficients are designated as,

$$\varepsilon = \frac{1}{2}(\gamma_{n\ell\ell} + \gamma_{m\bar{m}\ell}) = \frac{1}{2}\Big[\mathbf{e}^\nu_n(\nabla_\mu \mathbf{e}_{\ell\nu}) + \mathbf{e}^\nu_m(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\Big]\mathbf{e}^\mu_\ell = \frac{1}{2}\Big[n^\nu \nabla_\mu \ell_\nu + \bar{m}^\nu \nabla_\mu \bar{m}_\nu\Big]\ell^\mu, \tag{2.2.12a}$$

$$\gamma = \frac{1}{2}(\gamma_{n\ell n} + \gamma_{m\bar{m}n}) = \frac{1}{2}\Big[\mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\ell\nu}) + \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\Big]\mathbf{e}_n^\mu = \frac{1}{2}\Big[n^\nu\nabla_\mu\ell_\nu + \bar{m}^\nu\nabla_\mu\bar{m}_\nu\Big]n^\mu, \tag{2.2.12b}$$

$$\beta = \frac{1}{2}(\gamma_{n\ell m} + \gamma_{m\bar{m}m}) = \frac{1}{2}\Big[\mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\ell\nu}) + \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\Big]\mathbf{e}_m^\mu = \frac{1}{2}\Big[n^\nu\nabla_\mu\ell_\nu + \bar{m}^\nu\nabla_\mu\bar{m}_\nu\Big]m^\mu, \tag{2.2.12c}$$

$$\alpha = \frac{1}{2}(\gamma_{n\ell\bar{m}} + \gamma_{m\bar{m}\bar{m}}) = \frac{1}{2}\Big[\mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\ell\nu}) + \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\Big]\mathbf{e}_{\bar{m}}^\mu = \frac{1}{2}\Big[n^\nu\nabla_\mu\ell_\nu + \bar{m}^\nu\nabla_\mu\bar{m}_\nu\Big]\bar{m}^\mu. \tag{2.2.12d}$$

$$\kappa = \gamma_{m\ell\ell} = \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\ell\nu})\mathbf{e}_\ell^\mu = m^\nu\nabla_\mu\ell_\nu\ell^\mu, \tag{2.2.12e}$$

$$\tau = \gamma_{m\ell n} = \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\ell\nu})\mathbf{e}_n^\mu = m^\nu\nabla_\mu\ell_\nu n^\mu, \tag{2.2.12f}$$

$$\sigma = \gamma_{m\ell m} = \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\ell\nu})\mathbf{e}_m^\mu = m^\nu\nabla_\mu\ell_\nu m^\mu, \tag{2.2.12g}$$

$$\rho = \gamma_{m\ell\bar{m}} = \mathbf{e}_m^\nu(\nabla_\mu \mathbf{e}_{\ell\nu})\mathbf{e}_{\bar{m}}^\mu = m^\nu\nabla_\mu\ell_\nu \bar{m}^\mu. \tag{2.2.12h}$$

$$\pi = \gamma_{n\bar{m}\ell} = \mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\mathbf{e}_\ell^\mu = n^\nu\nabla_\mu\bar{m}_\nu\ell^\mu, \tag{2.2.12i}$$

$$\nu = \gamma_{n\bar{m}n} = \mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\mathbf{e}_n^\mu = n^\nu\nabla_\mu\bar{m}_\nu n^\mu, \tag{2.2.12j}$$

$$\mu = \gamma_{n\bar{m}m} = \mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\mathbf{e}_m^\mu = n^\nu\nabla_\mu\bar{m}_\nu m^\mu, \tag{2.2.12k}$$

$$\lambda = \gamma_{n\bar{m}\bar{m}} = \mathbf{e}_n^\nu(\nabla_\mu \mathbf{e}_{\bar{m}\nu})\mathbf{e}_{\bar{m}}^\mu = n^\nu\nabla_\mu\bar{m}_\nu \bar{m}^\mu. \tag{2.2.12l}$$

Through this, we can specify the nature of the directional derivatives through their actions on the basis vectors in terms of their components. Now, we expand,

$$\begin{aligned} D\boldsymbol{\ell} &= \ell^\mu\nabla_\mu\boldsymbol{\ell} = \ell^\mu\nabla_\mu(\ell^\nu\mathbf{e}_\nu) = -\gamma_{(a)\ell\ell}\mathbf{e}^{(a)} \\ &= \gamma_{n\ell\ell}\boldsymbol{\ell} - \gamma_{m\ell\ell}\bar{\boldsymbol{m}} - \gamma_{\bar{m}\ell\ell}\boldsymbol{m} \\ &= (\varepsilon + \bar{\varepsilon})\boldsymbol{\ell} - \bar{\kappa}\boldsymbol{m} - \kappa\bar{\boldsymbol{m}}. \end{aligned} \tag{2.2.13a}$$

Similarly, we can further note analogously, the concise representation in terms of the components of the directional derivatives,

$$D\boldsymbol{n} = -(\epsilon + \bar{\epsilon})\boldsymbol{n} + \pi\boldsymbol{m} + \bar{\pi}\bar{\boldsymbol{m}}, \tag{2.2.13b}$$

$$D\boldsymbol{m} = \bar{\pi}\boldsymbol{\ell} - \kappa\boldsymbol{n} + (\epsilon - \bar{\epsilon})\boldsymbol{m}, \tag{2.2.13c}$$

$$\Delta\boldsymbol{\ell} = (\gamma + \bar{\gamma})\boldsymbol{\ell} - \bar{\tau}\boldsymbol{m} - \tau\bar{\boldsymbol{m}}, \tag{2.2.13d}$$

$$\Delta\boldsymbol{n} = -(\gamma + \bar{\gamma})\boldsymbol{n} + \nu\boldsymbol{m} + \bar{\nu}\bar{\boldsymbol{m}}, \tag{2.2.13e}$$

$$\Delta\boldsymbol{m} = \bar{\nu}\boldsymbol{\ell} - \tau\boldsymbol{n} + (\gamma - \bar{\gamma})\boldsymbol{m}, \tag{2.2.13f}$$

$$\delta\boldsymbol{\ell} = (\bar{\alpha} + \beta)\boldsymbol{\ell} - \bar{\rho}\boldsymbol{m} - \sigma\bar{\boldsymbol{m}}, \tag{2.2.13g}$$

$$\delta\boldsymbol{n} = -(\bar{\alpha} + \beta)\boldsymbol{n} + \mu\boldsymbol{m} + \bar{\lambda}\bar{\boldsymbol{m}}, \tag{2.2.13h}$$

$$\delta \boldsymbol{m} = \bar{\lambda}\boldsymbol{\ell} - \sigma \boldsymbol{n} + (\beta - \bar{\alpha})\boldsymbol{m}, \tag{2.2.13i}$$

$$\bar{\delta} \boldsymbol{m} = \bar{\mu}\boldsymbol{\ell} - \rho \boldsymbol{n} + (\alpha - \bar{\beta})\boldsymbol{m}. \tag{2.2.13j}$$

A convenient method for encoding, through the exterior derivatives and Cartan's Identity as elaborated in Sec. A.1, as

$$\begin{aligned} d\boldsymbol{\ell} &= (\epsilon + \bar{\epsilon})\boldsymbol{\ell} \wedge \boldsymbol{n} + [\bar{\tau} - (\alpha + \bar{\beta})]\boldsymbol{\ell} \wedge \boldsymbol{m} + [\tau - (\bar{\alpha} + \beta)]\boldsymbol{\ell} \wedge \bar{\boldsymbol{m}} \\ &\quad + \bar{\kappa}\boldsymbol{n} \wedge \boldsymbol{m} + \kappa \boldsymbol{n} \wedge \bar{\boldsymbol{m}} + (\bar{\rho} - \rho)\boldsymbol{m} \wedge \bar{\boldsymbol{m}}, \end{aligned} \tag{2.2.14a}$$

$$\begin{aligned} d\boldsymbol{n} &= (\gamma + \bar{\gamma})\boldsymbol{\ell} \wedge \boldsymbol{n} - \nu \boldsymbol{\ell} \wedge \boldsymbol{m} - \bar{\nu}\boldsymbol{\ell} \wedge \bar{\boldsymbol{m}} - [\pi - (\alpha + \bar{\beta})]\boldsymbol{n} \wedge \boldsymbol{m} \\ &\quad - [\bar{\pi} - (\bar{\alpha} + \beta)]\boldsymbol{n} \wedge \bar{\boldsymbol{m}} + (\bar{\mu} - \mu)\boldsymbol{m} \wedge \bar{\boldsymbol{m}}, \end{aligned} \tag{2.2.14b}$$

$$\begin{aligned} d\boldsymbol{m} &= (\tau + \bar{\pi})\boldsymbol{\ell} \wedge \boldsymbol{n} - [\bar{\mu} + (\gamma - \bar{\gamma})]\boldsymbol{\ell} \wedge \boldsymbol{m} + \bar{\lambda}\boldsymbol{\ell} \wedge \bar{\boldsymbol{m}} \\ &\quad + [\rho - (\epsilon - \bar{\epsilon})]\boldsymbol{n} \wedge \boldsymbol{m} + \sigma \boldsymbol{n} \wedge \bar{\boldsymbol{m}} + (\bar{\alpha} - \beta)\boldsymbol{m} \wedge \bar{\boldsymbol{m}}. \end{aligned} \tag{2.2.14c}$$

2.2.2 Curvature

By our analogous representation, we can realise the the parallelism to the covariant and contravariant indices, such that we have the tetrad components of the Eq. (1.1.6), as

$$\begin{aligned} C_{(a)(b)(c)(d)} &= R_{(a)(b)(c)(d)} - \frac{1}{2}\Big(R_{(a)(c)}\eta_{(b)(d)} + R_{(b)(d)}\eta_{(a)(c)} - R_{(a)(d)}\eta_{(b)(c)} \\ &\quad - R_{(b)(c)}\eta_{(a)(d)}\Big) + \frac{1}{6}R\Big(\eta_{(a)(c)}\eta_{(b)(d)} - \eta_{(b)(c)}\eta_{(a)(d)}\Big). \end{aligned} \tag{2.2.15}$$

where R_{ac} denotes the tetrad components of the Ricci tensor and R is the scalar Ricci curvature given by

$$R_{(a)(c)} = \eta^{bd}R_{(a)(b)(c)(d)}, \tag{2.2.16a}$$

$$\begin{aligned} R &= \eta^{(a)(b)}R_{(a)(b)} = R_{\ell m} + R_{m\ell} - R_{m\bar{m}} - R_{\bar{m}m} \\ &= 2(R_{\ell n} - R_{m\bar{m}}). \end{aligned} \tag{2.2.16b}$$

where we have utilized the antisymmetry of the Ricci tensor.

Further, we have the trace-free nature translating in the projected tetrad frame, as,

$$\eta^{(a)(d)}C_{(a)(b)(c)(d)} = C_{\ell(b)(c)n} + C_{n(b)(c)\ell} - C_{m(b)(c)\bar{m}} - C_{\bar{m}(b)(c)m} = 0. \tag{2.2.17}$$

Besides, we must also require that, by the Bianchi identity translated into the tetrad frame as,

$$C_{\ell n m\bar{m}} + C_{\ell m\bar{m}n} + C_{\ell\bar{m}nm} = 0. \tag{2.2.18}$$

From the above Eq. (2.2.17), (2.2.18), we have the following constraints on the Weyl tensor components as

$$\begin{aligned} C_{\ell m\ell\bar{m}} &= -C_{m\ell\ell\bar{m}} = -C_{\ell\ell\ell n} - C_{n\ell\ell\ell} + C_{\bar{m}\ell\ell m} \\ &= C_{\bar{m}\ell\ell m} = C_{\ell m\bar{m}\ell} = -C_{\ell m\ell\bar{m}} = 0, \end{aligned} \tag{2.2.19a}$$

Similarly, we have

$$C_{nm\ell\bar{m}} = C_{m\ell\ell\bar{m}} = -C_{\ell mm\ell} - C_{\ell mm\ell} + C_{\bar{m}mmm} = -C_{\ell m\ell\bar{m}} = 0, \tag{2.2.19b}$$

$$C_{\ell mmn} = -C_{\ell mm\ell} + C_{mmmm} + C_{\bar{m}mmm} = -C_{\ell mm\ell} = -C_{\ell mm\ell} = 0, \tag{2.2.19c}$$

$$C_{\ell\bar{m}\bar{m}n} = -C_{\ell\bar{m}\bar{m}\ell} + C_{m\bar{m}\bar{m}\bar{m}} + C_{\bar{m}\bar{m}\bar{m}m} = -C_{\ell\bar{m}\bar{m}\ell} = -C_{\ell\bar{m}\bar{m}\ell} = 0, \tag{2.2.19d}$$

$$C_{\ell nm\ell} = -C_{\ell m\ell n} = C_{\ell m\ell\ell} - C_{mm\ell\bar{m}} - C_{\bar{m}m\ell m} = -C_{\bar{m}m\ell m} = C_{\ell mm\bar{m}}, \tag{2.2.19e}$$

$$C_{\ell n\bar{m}\ell} = -C_{\ell\bar{m}\ell n} = C_{\ell\bar{m}\ell\ell} - C_{m\bar{m}\ell\bar{m}} - C_{\bar{m}\bar{m}\ell m} = -C_{m\bar{m}\ell\bar{m}} = C_{\ell\bar{m}\bar{m}m}, \tag{2.2.19f}$$

$$C_{\ell nmn} = -C_{\ell\ell m\ell} + C_{m\ell m\bar{m}} + C_{\bar{m}\ell mm} = C_{m\ell m\bar{m}} = C_{\ell m\bar{m}m}, \tag{2.2.19g}$$

$$C_{\ell n\bar{m}n} = -C_{\ell\ell\bar{m}\ell} + C_{m\ell\bar{m}\bar{m}} + C_{\bar{m}\ell\bar{m}m} = C_{\bar{m}\ell\bar{m}m} = C_{\ell\bar{m}m\bar{m}}. \tag{2.2.19h}$$

Further, from the cyclic property in Eq. (2.2.18), we arrive at

$$\begin{aligned} C_{\ell n\ell n} &= -C_{nn\ell\ell} + C_{mn\ell\bar{m}} + C_{\bar{m}n\ell m} = C_{mn\ell\bar{m}} + C_{\bar{m}n\ell m} \\ &= (-C_{m\ell\bar{m}n} - C_{m\bar{m}n\ell}) + (-C_{\bar{m}\ell mn} - C_{\bar{m}mn\ell}) \\ &= C_{\ell m\bar{m}n} + C_{nm\bar{m}\ell} = C_{mm\bar{m}\bar{m}} + C_{m\bar{m}m\bar{m}} = C_{m\bar{m}m\bar{m}}, \end{aligned} \tag{2.2.19i}$$

$$\begin{aligned} C_{\ell m\bar{m}n} &= -C_{nm\bar{m}\ell} + C_{mm\bar{m}\bar{m}} + C_{\bar{m}m\bar{m}m} = C_{\ell\bar{m}nm} + C_{\bar{m}m\bar{m}m} \\ &= (-C_{\ell nm\bar{m}} - C_{\ell m\bar{m}n}) + C_{m\bar{m}m\bar{m}} = \frac{1}{2}(C_{m\bar{m}m\bar{m}} - C_{\ell nm\bar{m}}). \end{aligned} \tag{2.2.19j}$$

We now can relate various components of the Riemann tensor to those of the Weyl and the Ricci tensors as

$$\begin{aligned} R_{\ell n\ell n} &= C_{\ell n\ell n} + \frac{1}{2}(R_{\ell n} + R_{n\ell}) - \frac{1}{6}R \\ &= C_{\ell n\ell n} + R_{\ell n} - \frac{1}{6}R, \end{aligned} \tag{2.2.20a}$$

In a similar notion, we can further rewrite the other independent components as

$$\begin{aligned} R_{\ell mn\bar{m}} &= C_{\ell mn\bar{m}} + \frac{1}{2}(-R_{m\bar{m}} + R_{\ell n}) - \frac{1}{6}R \\ &= C_{\ell mn\bar{m}} + \frac{1}{2}\left(\frac{1}{2}R\right) - \frac{1}{6}R \\ &= C_{\ell mn\bar{m}} + \frac{1}{12}R, \end{aligned} \tag{2.2.20b}$$

$$R_{\ell nm\bar{m}} = C_{\ell nm\bar{m}}, \tag{2.2.20c}$$

$$R_{m\bar{m}m\bar{m}} = C_{m\bar{m}m\bar{m}} - \frac{1}{2}(R_{m\bar{m}} - R_{\bar{m}m}) - \frac{1}{6}R$$

$$= C_{m\bar{m}m\bar{m}} - \frac{1}{6}R, \tag{2.2.20d}$$

$$R_{\ell m\ell m} = C_{\ell m\ell m}, \tag{2.2.20e}$$

$$R_{nmnm} = C_{nmnm}, \tag{2.2.20f}$$

$$R_{\ell m\ell\bar{m}} = C_{\ell m\ell\bar{m}} + \frac{1}{2}R_{\ell\ell} = \frac{1}{2}R_{\ell\ell}, \tag{2.2.20g}$$

$$R_{nmn\bar{m}} = C_{nmn\bar{m}} + \frac{1}{2}R_{nn} = \frac{1}{2}R_{nn}, \tag{2.2.20h}$$

$$R_{m\ell mn} = C_{m\ell mn} - \frac{1}{2}R_{mm} = -\frac{1}{2}R_{mm}, \tag{2.2.20i}$$

$$R_{\ell n\ell m} = C_{\ell n\ell m} + \frac{1}{2}R_{\ell m}, \tag{2.2.20j}$$

$$R_{\ell mm\bar{m}} = C_{\ell mm\bar{m}} + \frac{1}{2}R_{\ell m}, \tag{2.2.20k}$$

$$R_{\ell nnm} = C_{\ell nnm} - \frac{1}{2}R_{nm}, \tag{2.2.20l}$$

$$R_{nmm\bar{m}} = C_{nmm\bar{m}} + \frac{1}{2}R_{nm}. \tag{2.2.20m}$$

Further, we combine the 10 independent components of the Weyl tensor in this formalism as five complex scalars, given by

$$\Psi_0 = C_{\ell m\ell m} = C_{\mu\nu\alpha\beta}\ell^\mu m^\nu \ell^\alpha m^\beta, \tag{2.2.21a}$$

$$\Psi_1 = C_{\ell n\ell m} = C_{\mu\nu\alpha\beta}\ell^i n^\nu \ell^\alpha m^\beta, \tag{2.2.21b}$$

$$\Psi_2 = C_{\ell m\bar{m}n} = C_{\mu\nu\alpha\beta}\ell^\mu m^\nu \bar{m}^\alpha n^\beta, \tag{2.2.21c}$$

$$\Psi_3 = C_{\ell n\bar{m}n} = C_{\mu\nu\alpha\beta}\ell^i n^\nu \bar{m}^\alpha n^\beta, \tag{2.2.21d}$$

$$\Psi_4 = C_{n\bar{m}n\bar{m}} = C_{\mu\nu\alpha\beta}n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta. \tag{2.2.21e}$$

The Weyl tensor can be independently written as a combination of the five complex scalars.

Finally, the ten components of the Ricci tensor are defined in terms of the following four real and three complex scalars as

$$\Lambda = \frac{R}{24} = \frac{1}{12}(R_{\ell n} - R_{m\bar{m}}) = \frac{1}{4}R_{\mu\nu}(\ell^p n^q - m^p \bar{m}^q), \tag{2.2.22a}$$

$$\Phi_{00} = \frac{1}{2}R_{\ell\ell} = \frac{1}{2}R_{\mu\nu}\ell^\mu \ell^\nu, \tag{2.2.22b}$$

$$\Phi_{11} = \frac{1}{4}(R_{\ell n} + R_{m\bar{m}}) = \frac{1}{4}R_{\mu\nu}(\ell^\mu n^\nu + m^\mu \bar{m}^\nu), \tag{2.2.22c}$$

$$\Phi_{22} = \frac{1}{2}R_{nn} = \frac{1}{2}R_{\mu\nu}n^\mu n^\nu, \tag{2.2.22d}$$

$$\Phi_{01} = \frac{1}{2}R_{\ell m} = \frac{1}{2}R_{\mu\nu}\ell^\mu m^\nu, \tag{2.2.22e}$$

$$\Phi_{10} = \bar{\Phi}_{01} = -\frac{1}{2}R_{\ell\bar{m}} = \frac{1}{2}R_{\mu\nu}\ell^{\mu}\bar{m}^{\nu}, \tag{2.2.22f}$$

$$\Phi_{02} = \frac{1}{2}R_{mm} = \frac{1}{2}R_{\mu\nu}m^{\mu}m^{\nu}, \tag{2.2.22g}$$

$$\Phi_{20} = \bar{\Phi}_{02} = -\frac{1}{2}R_{\bar{m}\bar{m}} = \frac{1}{2}R_{\mu\nu}\bar{m}^{\mu}\bar{m}^{\nu}, \tag{2.2.22h}$$

$$\Phi_{12} = \frac{1}{2}R_{nm} = \frac{1}{2}R_{\mu\nu}n^{\mu}m^{\nu}, \tag{2.2.22i}$$

$$\Phi_{21} = \bar{\Phi}_{12} = -\frac{1}{2}R_{n\bar{m}} = \frac{1}{2}R_{\mu\nu}n^{\mu}\bar{m}^{\nu}. \tag{2.2.22j}$$

2.2.3 Newman-Penrose Equations

Since the null tetrad is not typically a coordinate basis, we have some non-trivial commutation relations. We begin with explicitly stating commutation relations, with the formulation of a *Lie bracket* $[\boldsymbol{e}_{(a)}, \boldsymbol{e}_{(b)}]$, for a general tetrad, as

$$[\boldsymbol{e}_{(a)}, \boldsymbol{e}_{(b)}] = (\gamma_{(c)(b)(a)} - \gamma_{(c)(a)(b)})\boldsymbol{e}^{(c)} = C^{(c)}_{(a)(b)}\boldsymbol{e}_{(c)}. \tag{2.2.23}$$

For instance, we expand the relation for a scalar function f as

$$\begin{aligned}
[\Delta, D]f &= \Delta(Df) - D(\Delta f) = n^{\mu}\nabla_{\mu}(\ell^{\nu}\nabla_{\nu}f) - \ell^{\mu}\nabla_{\mu}(n^{\nu}\nabla_{\nu}f) \\
&= n^{\mu}\nabla_{\mu}\ell^{\nu}\nabla_{\nu}f + n^{\mu}\ell^{\nu}\nabla_{\mu}\nabla_{\nu}f - \ell^{\mu}\nabla_{i}n^{\nu}\nabla_{\nu}f - \ell^{i}n^{\nu}\nabla_{\mu}\nabla_{\nu}f \\
&= \left(n^{\mu}\nabla_{\mu}\ell^{\nu} - \ell^{\mu}\nabla_{i}n^{\nu}\right)\nabla_{\nu}f = (\Delta\ell^{\nu} - Dn^{\nu})\nabla_{\nu}f.
\end{aligned}$$

Thereby, we substitute the Ricci-rotation coefficients from Eq. (2.2.13) such that we simplify as

$$\begin{aligned}
[\Delta, D] &= (\Delta\ell^{\nu} - Dn^{\nu})\nabla_{\nu} \\
&= ((\gamma + \bar{\gamma})\ell^{\nu} - \bar{\tau}m^{\nu} - \tau\bar{m}^{\nu} + (\epsilon + \bar{\epsilon})n^{\nu} - \pi m^{\nu} - \bar{\pi}\bar{m}^{\nu})\nabla_{\nu} \\
&= (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}, \qquad (2.2.24a)
\end{aligned}$$

such that we obtain the requisite structure constants. In a similar fashion, we have

$$[\delta, D] = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}, \tag{2.2.24b}$$

$$[\delta, \Delta] = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}, \tag{2.2.24c}$$

$$[\bar{\delta}, \delta] = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}. \tag{2.2.24d}$$

Further, we shall explicitly compute components of Weyl tensor from Eq. (2.2.21), through the relations in Eq. (2.2.20), in terms of the rotation coefficients by substituting them in Eq. (2.1.26), beginning with,

$$\Psi_0 = C_{\ell n\ell n} = R_{\ell n\ell n}$$

$$
\begin{aligned}
&= -\partial_m \gamma_{\ell m \ell} + \partial_\ell \gamma_{\ell m m} + \gamma_{m \ell a}\left[\gamma^a_{\ell m} - \gamma^a_{m \ell}\right] + \gamma^a_{mm}\gamma_{a \ell \ell} - \gamma^a_{m\ell}\gamma_{a\ell m} \\
&= \partial_\ell \gamma_{\ell m m} - \partial_m \gamma_{\ell m \ell} - \gamma_{\ell m \ell}\left(\gamma_{\bar{m} m m} + \gamma_{\ell n m} - \gamma_{n \ell m} + \gamma_{n m \ell} + \gamma_{\ell m n}\right) \\
&\quad + \gamma_{\ell m m}\left(\gamma_{\bar{m} m m} + \gamma_{\ell n m} - \gamma_{n \ell m} + \gamma_{n m \ell} + \gamma_{\ell m n}\right) \\
&= D\sigma - \delta\kappa - \kappa\left(\bar{\pi} - \tau - 3\beta - \bar{\alpha}\right) - \sigma\left(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}\right)
\end{aligned} \tag{2.2.25a}
$$

We proceed further in a similar sense to derive other components of the Weyl tensor in Eq. (2.2.21) as,

$$
\begin{aligned}
\Psi_1 &= C_{\ell n \ell m} = R_{\ell m \ell n} - \frac{1}{2}R_{\ell m} \\
&= D\tau - \Delta\kappa - \rho\left(\tau + \bar{\pi}\right) - \sigma\left(\bar{\tau} + \pi\right) - \tau\left(\epsilon - \bar{\epsilon}\right) + \kappa\left(3\gamma + \bar{\gamma}\right) - \Phi_{01}, \quad (2.2.25b) \\
\Psi_2 &= C_{\ell m \bar{m} n} = R_{n m \bar{m}} - \frac{1}{12}R \\
&= D\mu - \delta\pi - \left(\bar{\rho}\mu + \sigma\lambda + \nu\kappa\right) - \pi\left(\bar{\pi} - \bar{\alpha} + \beta\right) + \mu\left(\epsilon + \bar{\epsilon}\right) - 2\Lambda, \quad (2.2.25c) \\
\Psi_3 &= C_{\ell n \bar{m} n} = C_{n \bar{m} m \bar{m}} = R_{n \bar{m} n \ell} + \frac{1}{2}R_{n \bar{m}} \\
&= D\nu - \Delta\pi - \mu\left(\pi + \bar{\tau}\right) - \lambda\left(\bar{\pi} + \tau\right) - \pi\left(\gamma - \bar{\gamma}\right) - \nu\left(3\epsilon + \bar{\epsilon}\right) - \Phi_{21}, \quad (2.2.25d) \\
\Psi_4 &= C_{n \bar{m} n \bar{m}} = R_{n \bar{m} n \bar{m}} \\
&= \bar{\delta}\nu - \Delta\lambda - \lambda\left(\mu + \bar{\mu} - 3\gamma - \bar{\gamma}\right) + \nu\left(3\alpha + \bar{\beta} + \pi - \bar{\tau}\right). \quad (2.2.25e)
\end{aligned}
$$

Further, as noted in the context of tetrad formulation, there are 36 equations of the kind Eq. (2.1.29), but we can condense the above equations and expand further components of the Riemann tensor and Ricci tensors, to arrive at the following 18 equations (and their complex conjugates). For ease of presentation, the equations containing the Δ operator will be radial, as follows,

$$
\begin{aligned}
\Delta\pi - D\nu &= -\mu\left(\pi + \bar{\tau}\right) - \lambda\left(\bar{\pi} + \tau\right) - \pi\left(\gamma - \bar{\gamma}\right) + \nu\left(3\epsilon + \bar{\epsilon}\right) - \Psi_3 - \Phi_{21}, \quad (2.2.26a) \\
\Delta\beta - \delta\gamma &= \gamma\left(\bar{\alpha} + \beta - \tau\right) - \mu\tau + \sigma\nu + \epsilon\bar{\nu} + \beta\left(\gamma - \bar{\gamma} - \mu\right) - \alpha\bar{\lambda} - \Phi_{12}, \quad (2.2.26b) \\
\Delta\alpha - \bar{\delta}\gamma &= \nu\left(\rho + \epsilon\right) - \lambda\left(\tau + \beta\right) + \alpha\left(\bar{\gamma} - \bar{\mu}\right) + \gamma\left(\bar{\beta} - \bar{\tau}\right) - \Psi_3, \quad (2.2.26c) \\
\Delta\lambda - \bar{\delta}\nu &= \lambda\left(\bar{\gamma} - 3\gamma - \mu - \bar{\mu}\right) + \nu\left(3\alpha + \bar{\beta} + \pi - \bar{\tau}\right) - \Psi_4, \quad (2.2.26d) \\
\Delta\mu - \delta\nu &= -\mu^2 - |\lambda|^2 - \mu\left(\gamma + \bar{\gamma}\right) + \bar{\nu}\pi + \nu\left(\bar{\alpha} + 3\beta - \tau\right) - \Phi_{22}, \quad (2.2.26e) \\
\Delta\rho - \bar{\delta}\tau &= -\rho\bar{\mu} - \sigma\lambda + \tau\left(\bar{\beta} - \alpha - \bar{\tau}\right) + \rho\left(\gamma + \bar{\gamma}\right) + \nu\kappa - \Psi_2 - 2\Lambda, \quad (2.2.26f) \\
\Delta\sigma - \delta\tau &= -\mu\sigma - \bar{\lambda}\rho + \tau\left(\tau + \beta - \bar{\alpha}\right) + \sigma\left(3\gamma - \bar{\gamma}\right) + \kappa\bar{\nu} - \Phi_{02}, \quad (2.2.26g) \\
\Delta\kappa - D\tau &= -\rho\left(\tau + \bar{\pi}\right) - \sigma\left(\bar{\tau} + \pi\right) - \tau\left(\epsilon - \bar{\epsilon}\right) + \kappa\left(3\gamma + \bar{\gamma}\right) - \Psi_1 - \Phi_{01}, \\
&\qquad (2.2.26h) \\
\Delta\epsilon - D\gamma &= -\alpha\left(\tau + \bar{\pi}\right) - \beta\left(\bar{\tau} + \pi\right) + 2\gamma\epsilon + \gamma\bar{\epsilon} + \epsilon\bar{\gamma} - \tau\pi + \nu\kappa - \Psi_2 - \Phi_{11} + \Lambda, \\
&\qquad (2.2.26i)
\end{aligned}
$$

with the evolution equations containing the directional derivative D, are,

$$
D\rho - \bar{\delta}\kappa = \rho^2 + |\sigma|^2 + \rho\left(\epsilon + \bar{\epsilon}\right) - \bar{\kappa}\tau + \kappa\left(\pi - 3\alpha - \bar{\beta}\right) + \Phi_{00}, \tag{2.2.26j}
$$

$$D\sigma-\delta\kappa=\sigma\left(3\epsilon-\bar{\epsilon}+\rho+\bar{\rho}\right)-\kappa\left(\bar{\pi}+\tau+3\bar{\beta}+\alpha\right)+\Psi_0, \tag{2.2.26k}$$
$$D\alpha-\bar{\delta}\epsilon=\alpha\left(\rho+\bar{\epsilon}-2\epsilon\right)+\beta\bar{\sigma}-\bar{\beta}\epsilon-\kappa\lambda-\bar{\kappa}\gamma+\pi\left(\epsilon+\rho\right)+\Phi_{10}, \tag{2.2.26l}$$
$$D\beta-\delta\epsilon=\sigma\left(\alpha+\pi\right)+\beta\left(\bar{\rho}-\bar{\epsilon}\right)-\kappa\left(\mu+\gamma\right)+\epsilon\left(\bar{\pi}-\bar{\alpha}\right)+\Psi_1, \tag{2.2.26m}$$
$$D\lambda-\bar{\delta}\pi=\rho\lambda+\bar{\sigma}\mu+\pi\left(\pi+\alpha-\bar{\beta}\right)-\nu\bar{\kappa}+\lambda\left(\bar{\epsilon}-3\epsilon\right)+\Phi_{20}, \tag{2.2.26n}$$
$$D\mu-\delta\pi=\bar{\rho}\mu+\sigma\lambda+\pi\left(\bar{\pi}-\bar{\alpha}+\beta\right)-\mu\left(\epsilon+\bar{\epsilon}\right)-\nu\kappa+\Psi_2+2\Lambda, \tag{2.2.26o}$$

and the angular field equations comprising the 2-surface derivative δ, as,

$$\delta\alpha-\bar{\delta}\beta=\mu\rho-\lambda\sigma+|\alpha|^2+|\beta|^2-2\alpha\beta+\gamma\left(\rho-\bar{\rho}\right)+\epsilon\left(\mu-\bar{\mu}\right)-\Psi_2+\Phi_{11}+\Lambda, \tag{2.2.26p}$$
$$\delta\rho-\bar{\delta}\sigma=\rho\left(\bar{\alpha}+\beta\right)+\sigma\left(\bar{\beta}-3\alpha\right)+\tau\left(\rho-\bar{\rho}\right)+\kappa\left(\mu-\bar{\mu}\right)-\Psi_1+\Phi_{01}, \tag{2.2.26q}$$
$$\delta\lambda-\bar{\delta}\mu=\nu\left(\rho-\bar{\rho}\right)+\pi\left(\mu-\bar{\mu}\right)+\mu\left(\alpha+\bar{\beta}\right)+\lambda\left(\bar{\alpha}-3\beta\right)-\Psi_3+\Phi_{21}. \tag{2.2.26r}$$

Further, the projection of the Riemann tensor covariant derivative with cyclic exchange of indices leads to the first Bianchi identities,

$$\begin{aligned}D\Psi_2-\bar{\delta}\Psi_1-\bar{\delta}\Phi_{01}+\Delta\Phi_{00}=&-\lambda\Psi_0+2\Psi_1\left(\pi-\alpha\right)+3\rho\Psi_2-2\kappa\Psi_3-2\Phi_{01}\left(\alpha+\bar{\tau}\right)\\&+2\rho\Phi_{11}+\bar{\sigma}\Phi_{02}-\Phi_{00}\left(\bar{\mu}-2\gamma-2\bar{\gamma}\right)-2\tau\Phi_{10}\\&-2D\Lambda,\end{aligned} \tag{2.2.27a}$$
$$\begin{aligned}\bar{\delta}\Psi_2-D\Psi_3-\delta\Phi_{20}+D\Phi_{21}=&\,2\lambda\Psi_1-3\pi\Psi_2+2\Psi_3\left(\epsilon-\rho\right)+\kappa\Psi_4+2\Phi_{21}\left(\bar{\rho}-\epsilon\right)\\&-2\mu\Phi_{10}+2\pi\Phi_{11}-\bar{\kappa}\Phi_{22}-\Phi_{20}\left(2\bar{\alpha}-2\beta-\bar{\pi}\right)\\&-2\bar{\delta}\Lambda,\end{aligned} \tag{2.2.27b}$$
$$\begin{aligned}\Delta\Psi_1-\delta\Psi_2+\bar{\delta}\Phi_{02}-\Delta\Phi_{01}=&\,\nu\Psi_0+2\Psi_1\left(\gamma-\mu\right)-3\tau\Psi_2+2\sigma\Psi_3+2\Phi_{01}\left(\bar{\mu}-\gamma\right)\\&-2\rho\Phi_{12}-\bar{\nu}\Phi_{00}+2\tau\Phi_{11}+\Phi_{02}\left(\bar{\tau}-2\bar{\beta}+2\alpha\right)\\&+2\delta\Lambda,\end{aligned} \tag{2.2.27c}$$
$$\begin{aligned}\bar{\delta}\Psi_0-D\Psi_1-\delta\Phi_{00}+D\Phi_{01}=&\,\Psi_0\left(4\alpha-\pi\right)-2\Psi_1\left(2\rho+\epsilon\right)+3\kappa\Psi_2+2\Phi_{01}\left(\epsilon+\bar{\rho}\right)\\&+2\sigma\Phi_{10}-2\kappa\Phi_{11}-\bar{\kappa}\Phi_{02},\end{aligned} \tag{2.2.27d}$$
$$\begin{aligned}\Delta\Psi_2-\delta\Psi_3+D\Phi_{22}-\delta\Phi_{21}=&\,2\nu\Psi_1-3\mu\Psi_2+2\Psi_3\left(\beta-\tau\right)+\sigma\Psi_4+2\Phi_{21}\left(\bar{\pi}+\beta\right)\\&-2\mu\Phi_{11}-\bar{\lambda}\Phi_{20}+2\pi\Phi_{12}+\Phi_{22}\left(\bar{\rho}-2\epsilon-2\bar{\epsilon}\right)\\&-2\Delta\Lambda,\end{aligned} \tag{2.2.27e}$$
$$\begin{aligned}D\Psi_4-\bar{\delta}\Psi_3-\bar{\delta}\Phi_{21}+\Delta\Phi_{20}=&-3\lambda\Psi_2+2\Psi_3\left(2\pi+\alpha\right)-\Psi_4\left(4\epsilon-\rho\right)+2\Phi_{21}\left(\alpha-\bar{\tau}\right)\\&+2\nu\Phi_{10}+\bar{\sigma}\Phi_{22}-2\lambda\Phi_{11}-\Phi_{20}\left(\bar{\mu}+2\gamma-2\bar{\gamma}\right),\end{aligned} \tag{2.2.27f}$$
$$\begin{aligned}\Delta\Psi_0-\delta\Psi_1-\delta\Phi_{01}+D\Phi_{02}=&\,\Psi_0\left(4\gamma-\mu\right)-2\Psi_1\left(2\tau+\beta\right)+3\sigma\Psi_2+2\Phi_{01}\left(\bar{\pi}-\beta\right)\\&-2\kappa\Phi_{12}-\bar{\lambda}\Phi_{00}+2\sigma\Phi_{11}+\Phi_{02}\left(\bar{\rho}+2\epsilon-2\bar{\epsilon}\right),\end{aligned} \tag{2.2.27g}$$
$$\Delta\Psi_3-\delta\Psi_4-\Delta\Phi_{21}+\bar{\delta}\Phi_{22}=3\nu\Psi_2-2\Psi_3\left(\gamma+2\mu\right)-\Psi_4\left(\tau-4\beta\right)+2\Phi_{21}\left(\bar{\mu}+\gamma\right)$$

$$- 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + \Phi_{22}\left(\bar{\tau} - 2\alpha - 2\bar{\beta}\right), \tag{2.2.27h}$$

and the contraction gives the second Bianchi identities, namely

$$\begin{aligned}\bar{\delta}\Phi_{01} + \delta\Phi_{10} - D\left(\Phi_{11} + 3\Lambda\right) - \Delta\Phi_{00} = {}& \bar{\kappa}\Phi_{12} + \kappa\Phi_{21} + \Phi_{01}\left(2\alpha + 2\bar{\tau} - \pi\right) \\ & + \Phi_{10}\left(2\bar{\alpha} + 2\tau - \bar{\pi}\right) - 2\Phi_{11}\left(\rho + \bar{\rho}\right) \\ & - \bar{\sigma}\Phi_{02} - \sigma\Phi_{20} + \Phi_{00}\left(\mu + \bar{\mu} - 2\gamma - 2\bar{\gamma}\right),\end{aligned} \tag{2.2.27i}$$

$$\begin{aligned}\bar{\delta}\Phi_{12} + \delta\Phi_{21} - \Delta\left(\Phi_{11} + 3\Lambda\right) - D\Phi_{22} = {}& -\nu\Phi_{01} - \bar{\nu}\Phi_{10} + \Phi_{12}\left(\bar{\tau} - 2\bar{\beta} - 2\pi\right) \\ & + \Phi_{21}\left(\tau - 2\beta - 2\bar{\pi}\right) + 2\Phi_{11}\left(\mu + \bar{\mu}\right) \\ & - \Phi_{22}\left(\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon}\right) + \lambda\Phi_{02} + \bar{\lambda}\Phi_{20},\end{aligned} \tag{2.2.27j}$$

$$\begin{aligned}\delta\left(\Phi_{11} - 3\Lambda\right) - D\Phi_{12} - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} = {}& \kappa\Phi_{22} - \bar{\nu}\Phi_{00} + \Phi_{02}\left(\bar{\tau} - \pi + 2\alpha - 2\bar{\beta}\right) \\ & - \sigma\Phi_{21} + \bar{\lambda}\Phi_{10} + 2\Phi_{11}\left(\tau - \bar{\pi}\right) \\ & - \Phi_{12}\left(2\rho + \bar{\rho} - 2\bar{\epsilon}\right) + \Phi_{01}\left(2\bar{\mu} + \mu - 2\gamma\right).\end{aligned} \tag{2.2.27k}$$

Thereby, the basic equations of the Newman-Penrose formalism are comprised in the commutation relations in Eq. (2.2.23), the field equations in Eq. (2.2.26) derived from the Ricci identity, and the Bianchi identities in Eq. (2.2.27). In a vacuum spacetime, the Ricci scalars vanish, and the relevant relations can be further simplified.

2.2.4 Maxwell's Equations

The antisymmetric Maxwell field tensor $\boldsymbol{F}$ can be encoded into the following three complex scalars in the Newman-Penrose formalism, as projections given by,

$$\phi_0 = F_{\ell m} = F_{\mu\nu}\ell^\mu m^\nu, \tag{2.2.28a}$$

$$\phi_1 = \frac{1}{2}\left(F_{\ell n} + F_{\bar{m}m}\right) = \frac{1}{2}F_{\mu\nu}\left(\ell^\mu n^\nu + \bar{m}^\mu m^\nu\right), \tag{2.2.28b}$$

$$\phi_2 = F_{\bar{m}n} = F_{\mu\nu}\bar{m}^\mu n^\nu, \tag{2.2.28c}$$

where the qualitative behavior of the field tensor is absorbed into the *radiative* behavior through ϕ_0 and ϕ_2, and the *Coulombic* nature captured through ϕ_1.

The field tensor can be rewritten in terms of the null tetrad using the projected quantities, as,

$$F_{\mu\nu} = -2\,\mathrm{Re}\left\{\phi_0 n_{[\mu}\bar{m}_{\nu]} + \phi_1\left(\ell_{[\mu}n_{\nu]} + m_{[\mu}\bar{m}_{\nu]}\right) + \phi_2\ell_{[\mu}m_{\nu]}\right\}. \tag{2.2.28d}$$

The energy-momentum tensor for the electromagnetic field can be given in the tetrad basis as,

$$T_{(a)(b)} = \eta^{(c)(d)} F_{(a)(c)} F_{(b)(d)} - \frac{1}{4} F_{(c)(d)} F^{(c)(d)} \eta_{(a)(b)}, \tag{2.2.29a}$$

which can be written in terms of Maxwell's scalars as,

$$|\phi_0|^2 = \frac{1}{2} T_{\ell\ell}, \quad |\phi_1|^2 = \frac{1}{4}\left(T_{\ell n} + T_{m\bar{m}}\right), \quad |\phi_2|^2 = \frac{1}{2} T_{nn}, \tag{2.2.29b}$$

$$\phi_0 \bar{\phi}_1 = \frac{1}{2} T_{\ell m}, \quad \phi_1 \bar{\phi}_2 = \frac{1}{2} T_{nm}, \quad \phi_0 \bar{\phi}_2 = \frac{1}{2} T_{mm}. \tag{2.2.29c}$$

We further obtain the Newman-Penrose Maxwell equations, which are the different components of the general relativistic Maxwell equations in spinorial form, by the analogous nature of the intrinsic derivatives, as

$$\nabla_{[(a)} F_{(b)(c)]} = 0, \tag{2.2.30a}$$

$$\eta^{(c)(a)} \nabla_{(c)} F_{(a)(b)} = 4\pi J_{(b)}. \tag{2.2.30b}$$

The above quantities can be further reduced to the following equations, in terms of the projections of the antisymmetric Maxwell field tensor,

$$\nabla_\ell \phi_1 - \nabla_{\bar{m}} \phi_0 = 2\pi J_\ell, \tag{2.2.31a}$$

$$\nabla_m \phi_2 - \nabla_n \phi_1 = 2\pi J_n, \tag{2.2.31b}$$

$$\nabla_\ell \phi_2 - \nabla_{\bar{m}} \phi_1 = 2\pi J_m, \tag{2.2.31c}$$

$$\nabla_m \phi_1 - \nabla_n \phi_0 = 2\pi J_{\bar{m}}. \tag{2.2.31d}$$

In terms of the directional derivatives, the above equations can be expanded as,

$$D\phi_1 - \bar{\delta}\phi_0 = \phi_0 (\pi - 2\alpha) + 2\rho\phi_1 - \kappa\phi_2 + 2\pi J_\ell, \tag{2.2.32a}$$

$$\delta\phi_2 - \Delta\phi_1 = \phi_2 (\tau - 2\beta) - \nu\phi_0 + 2\mu\phi_1 + 2\pi J_n, \tag{2.2.32b}$$

$$\delta\phi_1 - \Delta\phi_0 = \phi_0 (\mu - 2\gamma) + 2\tau\phi_1 - \sigma\phi_2 + 2\pi J_m, \tag{2.2.32c}$$

$$D\phi_2 - \bar{\delta}\phi_1 = \phi_2 (\rho - 2\epsilon) - \lambda\phi_0 + 2\pi\phi_1 + 2\pi J_{\bar{m}}. \tag{2.2.32d}$$

We can further simplify the Ricci scalar components in Eq. (2.2.22) by realizing Einstein's equation in Eq. (1.1.7) and connecting the energy-momentum tensor to the Ricci scalars, in terms of the projections. Due to the trace-free nature, we have $\Lambda = 0$ and the Ricci tensor components are simplified from Einstein's equation,

$$\Phi_{\mu\nu} = 2\phi_\mu \bar{\phi}_\nu. \tag{2.2.33}$$

Substituting the appropriate Ricci scalars, the Newman-Penrose field equations in Eq. (2.2.26) simplify, with the radial equations,

$$\Delta\pi - D\nu = -\mu(\pi+\bar{\tau}) - \lambda(\bar{\pi}+\tau) - \pi(\gamma-\bar{\gamma}) + \nu(3\epsilon+\bar{\epsilon}) - \Psi_3 - 2\phi_2\bar{\phi}_1, \tag{2.2.34a}$$

$$\Delta\beta - \delta\gamma = \gamma(\bar{\alpha}+\beta-\tau) - \mu\tau + \sigma\nu + \epsilon\bar{\nu} + \beta(\gamma-\bar{\gamma}-\mu) - \alpha\bar{\lambda} - 2\phi_1\bar{\phi}_2, \tag{2.2.34b}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho+\epsilon) - \lambda(\tau+\beta) + \alpha(\bar{\gamma}-\bar{\mu}) + \gamma\left(\bar{\beta}-\bar{\tau}\right) - \Psi_3, \tag{2.2.34c}$$

$$\Delta\lambda - \bar{\delta}\nu = \lambda(\bar{\gamma}-3\gamma-\mu-\bar{\mu}) + \nu\left(3\alpha+\bar{\beta}+\pi-\bar{\tau}\right) - \Psi_4, \tag{2.2.34d}$$

$$\Delta\mu - \delta\nu = -\mu^2 - |\lambda|^2 - \mu(\gamma+\bar{\gamma}) + \bar{\nu}\pi + \nu(\bar{\alpha}+3\beta-\tau) - 2|\phi_2|^2, \tag{2.2.34e}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \sigma\lambda + \tau\left(\bar{\beta}-\alpha-\bar{\tau}\right) + \rho(\gamma+\bar{\gamma}) + \nu\kappa - \Psi_2, \tag{2.2.34f}$$

$$\Delta\sigma - \delta\tau = -\mu\sigma - \bar{\lambda}\rho + \tau(\tau+\beta-\bar{\alpha}) + \sigma(3\gamma-\bar{\gamma}) + \kappa\bar{\nu} - 2\phi_0\bar{\phi}_2, \tag{2.2.34g}$$

$$\Delta\kappa - D\tau = -\rho(\tau+\bar{\pi}) - \sigma(\bar{\tau}+\pi) - \tau(\epsilon-\bar{\epsilon}) + \kappa(3\gamma+\bar{\gamma}) - \Psi_1 - 2\phi_0\bar{\phi}_1, \tag{2.2.34h}$$

$$\Delta\epsilon - D\gamma = -\alpha(\tau+\bar{\pi}) - \beta(\bar{\tau}+\pi) + 2\gamma\epsilon + \gamma\bar{\epsilon} + \epsilon\bar{\gamma} - \tau\pi + \nu\kappa - \Psi_2 - 2|\phi_1|^2, \tag{2.2.34i}$$

with the evolution equations containing the directional derivative D, are,

$$D\rho - \bar{\delta}\kappa = \rho^2 + |\sigma|^2 + \rho(\epsilon+\bar{\epsilon}) - \bar{\kappa}\tau + \kappa\left(\pi-3\alpha-\bar{\beta}\right) + 2|\phi_0|^2, \tag{2.2.34j}$$

$$D\sigma - \delta\kappa = \sigma(3\epsilon-\bar{\epsilon}+\rho+\bar{\rho}) - \kappa(\bar{\pi}+\tau+3\beta+\alpha) + \Psi_0, \tag{2.2.34k}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho+\bar{\epsilon}-2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon+\rho) + 2\phi_1\bar{\phi}_0, \tag{2.2.34l}$$

$$D\beta - \delta\epsilon = \sigma(\alpha+\pi) + \beta(\bar{\rho}-\bar{\epsilon}) - \kappa(\mu+\gamma) + \epsilon(\bar{\pi}-\bar{\alpha}) + \Psi_1, \tag{2.2.34m}$$

$$D\lambda - \bar{\delta}\pi = \rho\lambda + \bar{\sigma}\mu + \pi(\pi+\alpha-\beta) - \nu\bar{\kappa} + \lambda(\bar{\epsilon}-3\epsilon) + 2\phi_2\bar{\phi}_0, \tag{2.2.34n}$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \sigma\lambda + \pi(\bar{\pi}-\bar{\alpha}+\beta) - \mu(\epsilon+\bar{\epsilon}) - \nu\kappa + \Psi_2, \tag{2.2.34o}$$

and the angular field equations comprising the 2-surface derivative δ, as,

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + |\alpha|^2 + |\beta|^2 - 2\alpha\beta + \gamma(\rho-\bar{\rho}) + \epsilon(\mu-\bar{\mu}) - \Psi_2 + 2|\phi_1|^2, \tag{2.2.34p}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha}+\beta) + \sigma\left(\bar{\beta}-3\alpha\right) + \tau(\rho-\bar{\rho}) + \kappa(\mu-\bar{\mu}) - \Psi_1 + 2\phi_0\bar{\phi}_1, \tag{2.2.34q}$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho-\bar{\rho}) + \pi(\mu-\bar{\mu}) + \mu\left(\alpha+\bar{\beta}\right) + \lambda(\bar{\alpha}-3\beta) - \Psi_3 + 2\phi_2\bar{\phi}_1. \tag{2.2.34r}$$

Further, the Bianchi identities in Eq. (2.2.27) simplify extensively by substituting the appropriate Ricci scalars, and using the Maxwell's equations as,

$$D\Psi_2 - \bar{\delta}\Psi_1 - 2\bar{\phi}_1\bar{\delta}\phi_0 + 2\bar{\phi}_0\Delta\phi_0 = -\lambda\Psi_0 + 2\Psi_1(\pi-\alpha) + 3\rho\Psi_2 - 2\kappa\Psi_3$$

$$- 4\alpha\bar{\phi}_1\phi_0 + 4\rho|\phi_1|^2 + 4\gamma|\phi_0|^2 - 4\tau\bar{\phi}_0\phi_1, \tag{2.2.35a}$$

$$\begin{aligned}\bar{\delta}\Psi_2 - D\Psi_3 - 2\bar{\phi}_0\delta\phi_2 + 2\bar{\phi}_1 D\phi_2 = {}& 2\lambda\Psi_1 - 3\pi\Psi_2 + 2\Psi_3(\epsilon-\rho) + \kappa\Psi_4 \\ & - 4\epsilon\bar{\phi}_1\phi_2 - 4\mu\bar{\phi}_0\phi_1 + 4\beta\bar{\phi}_0\phi_2 + 4\pi|\phi_1|^2,\end{aligned} \tag{2.2.35b}$$

$$\begin{aligned}\Delta\Psi_1 - \delta\Psi_2 + 2\bar{\phi}_2\bar{\delta}\phi_0 - 2\bar{\phi}_1\Delta\phi_0 = {}& \nu\Psi_0 + 2\Psi_1(\gamma-\mu) - 3\tau\Psi_2 + 2\sigma\Psi_3 \\ & - 4\rho\bar{\phi}_2\phi_1 - 4\gamma\bar{\phi}_1\phi_0 + 4\tau|\phi_1|^2 + 4\alpha\bar{\phi}_2\phi_0,\end{aligned} \tag{2.2.35c}$$

$$\begin{aligned}\bar{\delta}\Psi_0 - D\Psi_1 - 2\bar{\phi}_0\delta\phi_0 + 2\bar{\phi}_1 D\phi_0 = {}& \Psi_0(4\alpha-\pi) - 2\Psi_1(2\rho+\epsilon) + 3\kappa\Psi_2 \\ & + 4\epsilon|\phi_1|^2 + 4\sigma\bar{\phi}_0\phi_1 - 4\kappa|\phi_1|^2 - 4\beta|\phi_0|^2,\end{aligned} \tag{2.2.35d}$$

$$\begin{aligned}\Delta\Psi_2 - \delta\Psi_3 + 2\bar{\phi}_2 D\phi_2 - 2\bar{\phi}_1\delta\phi_2 = {}& 2\nu\Psi_1 - 3\mu\Psi_2 + 2\Psi_3(\beta-\tau) + \sigma\Psi_4 \\ & - 4\epsilon|\phi_2|^2 - 4\mu|\phi_1|^2 + 4\beta\bar{\phi}_1\phi_2 + 4\pi\bar{\phi}_2\phi_1,\end{aligned} \tag{2.2.35e}$$

$$\begin{aligned}D\Psi_4 - \bar{\delta}\Psi_3 - 2\bar{\phi}_1\bar{\delta}\phi_2 + 2\bar{\phi}_0\Delta\phi_2 = {}& -3\lambda\Psi_2 + 2\Psi_3(2\pi+\alpha) - \Psi_4(4\epsilon-\rho) \\ & + 4\alpha\bar{\phi}_1\phi_2 + 4\nu\bar{\phi}_0\phi_1 - 4\gamma\bar{\phi}_0\phi_2 - 4\lambda|\phi_1|^2,\end{aligned} \tag{2.2.35f}$$

$$\begin{aligned}\Delta\Psi_0 - \delta\Psi_1 - 2\bar{\phi}_1\delta\phi_0 + 2\bar{\phi}_2 D\phi_0 = {}& \Psi_0(4\gamma-\mu) - 2\Psi_1(2\tau+\beta) + 3\sigma\Psi_2 \\ & - 4\kappa\bar{\phi}_2\phi_1 - 4\beta\bar{\phi}_1\phi_0 + 4\sigma|\phi_1|^2 + 4\epsilon\bar{\phi}_2\phi_0,\end{aligned} \tag{2.2.35g}$$

$$\begin{aligned}\Delta\Psi_3 - \delta\Psi_4 - 2\bar{\phi}_1\Delta\phi_2 + 2\bar{\phi}_2\bar{\delta}\phi_2 = {}& 3\nu\Psi_2 - 2\Psi_3(\gamma+2\mu) - \Psi_4(\tau-4\beta) \\ & - 4\alpha|\phi_2|^2 - 4\nu|\phi_1|^2 + 4\gamma\bar{\phi}_1\phi_2 + 4\lambda\bar{\phi}_2\phi_1.\end{aligned} \tag{2.2.35h}$$

Since the vanishing of the covariant divergence of the energy-momentum tensor is ensured by Maxwell's equations, the contracted Bianchi identities do not provide any new information. Explicitly, they are given by

$$\begin{aligned}\bar{\delta}(\phi_0\bar{\phi}_1) + \delta(\phi_1\bar{\phi}_0) - D|\phi_1|^2 - \Delta|\phi_0|^2 = {}& \bar{\kappa}\phi_1\bar{\phi}_2 + \kappa\phi_2\bar{\phi}_1 + \phi_0\bar{\phi}_1(2\alpha + 2\bar{\tau} \\ & -\pi) + \phi_1\bar{\phi}_0(2\bar{\alpha} + 2\tau - \bar{\pi}) \\ & + 2|\phi_1|^2(\rho+\bar{\rho}) - \bar{\sigma}\phi_0\bar{\phi}_2 - \sigma\phi_2\bar{\phi}_0 \\ & + |\phi_0|^2(\mu + \bar{\mu} - 2\gamma - 2\bar{\gamma}),\end{aligned} \tag{2.2.35i}$$

$$\begin{aligned}\bar{\delta}(\phi_1\bar{\phi}_2) + \delta(\phi_2\bar{\phi}_1) - \Delta|\phi_1|^2 - D|\phi_2|^2 = {}& -\nu\phi_0\bar{\phi}_1 - \bar{\nu}\phi_1\bar{\phi}_0 + \phi_1\bar{\phi}_2\Big(\bar{\tau} - 2\bar{\beta} \\ & -2\pi) + \phi_2\bar{\phi}_1(\tau - 2\beta - 2\bar{\pi}) \\ & + 2|\phi_1|^2(\mu+\bar{\mu}) + \lambda\phi_0\bar{\phi}_2 + \bar{\lambda}\phi_2\bar{\phi}_0 \\ & - |\phi_2|^2(\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon}),\end{aligned} \tag{2.2.35j}$$

$$\delta|\phi_1|^2 - D(\phi_1\bar{\phi}_2) - \Delta(\phi_0\bar{\phi}_1) + \bar{\delta}(\phi_0\bar{\phi}_2) = \kappa|\phi_2|^2 - \bar{\nu}|\phi_0|^2 + \phi_0\bar{\phi}_2(\bar{\tau} - \pi$$

$$
\begin{aligned}
&+2\alpha-2\bar{\beta}\big)-\sigma\phi_2\bar{\phi}_1+\bar{\lambda}\phi_1\bar{\phi}_0\\
&+2|\phi_1|^2(\tau-\bar{\pi})-\phi_1\bar{\phi}_2(2\rho+\bar{\rho}\\
&-2\bar{\epsilon})+\phi_0\bar{\phi}_1(2\bar{\mu}+\mu-2\gamma).
\end{aligned} \tag{2.2.35k}
$$

2.3 Tetrad Transformations and Symmetries

2.3.1 Lorentz Transformations

The notion of the tetrad frame, being non-unique, can be subjected to transformations, including coordinate transformations $x^\mu \to x'^\mu$, which only affects the spacetime indices of the tetrad, given as

$$e^i_{(a)}(\boldsymbol{x}) \to e'^i_{(a)}(\boldsymbol{x}') = \frac{\partial x'^\mu}{\partial x^\nu} e^j_{(a)}(\boldsymbol{x}). \tag{2.3.1a}$$

Further, since the tetrad basis is a vectorial basis, it transforms as a vector in the tetrad index (a), as

$$e^i_{(a)}(\boldsymbol{x}) \to e^i_{(a')}(\boldsymbol{x}) = \Lambda^{(b)}_{(a')} e^i_{(b)}(\boldsymbol{x}) \tag{2.3.1b}$$

where $\Lambda^{(b)}_{(a')}$ is a rigid transformation of the tetrad basis, which preserves the orthonormality condition in Eq. (2.1.5). Thereby, we have,

$$
\begin{aligned}
\eta_{(a)(b)} = e^\mu_{(a)} e_{i(b)} &= \Big(\Lambda^{(a')}_{(a)} e^\mu_{(a')}\Big)\Big(\Lambda^{(b')}_{(b)} e_{i(b')}\Big)\\
&= \Lambda^{(a')}_{(a)}\Lambda^{(b')}_{(b)} e^\mu_{(a')} e_{i(b')} = \Lambda^{(a')}_{(a)}\Lambda^{(b')}_{(b)} \eta_{(a')(b')},
\end{aligned} \tag{2.3.1c}
$$

thus, for the invariance of $\eta_{(a)(b)}$, we need Λ to necessarily a Lorentz transformation. These Lorentz transformations preserve the inner product of the tetrad components and can be applied consistently across spacetime.

Corresponding to the six parameters of the group of Lorentz transformations, we have six degrees of freedom to rotate a chosen tetrad frame. In considering the effect of such Lorentz transformations on the various Newman-Penrose quantities, we shall find it convenient to regard a general Lorentz transformation of the basis vectors, which can be grouped as following classes:

Class I: Null rotations about $\boldsymbol{\ell}$

This transformation leaves $\boldsymbol{\ell}$ invariant, while modifying $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, such that for a complex parameter, $a \in \mathbb{C}$, we have

$$\boldsymbol{\ell} \to \boldsymbol{\ell}, \qquad \boldsymbol{n} \to \boldsymbol{n} + \bar{a}\boldsymbol{m} + a\bar{\boldsymbol{m}} + |a|^2\boldsymbol{\ell}, \qquad \boldsymbol{m} \to \boldsymbol{m} + a\boldsymbol{\ell} \tag{2.3.2}$$

We can similarly compute the corresponding changes in the spin coefficients through the directional derivatives as,

$$\kappa \to \kappa, \tag{2.3.3a}$$
$$\sigma \to \sigma + a\kappa, \tag{2.3.3b}$$
$$\rho \to \rho + \bar{a}\kappa, \tag{2.3.3c}$$
$$\epsilon \to \epsilon + \bar{a}\kappa, \tag{2.3.3d}$$
$$\tau \to \tau + a\rho + \bar{a}\sigma + |a|^2\kappa, \tag{2.3.3e}$$
$$\pi \to \pi + 2\bar{a}\epsilon + \bar{a}^2\kappa + D\bar{a}, \tag{2.3.3f}$$
$$\alpha \to \alpha + \bar{a}(\rho + \epsilon) + \bar{a}^2\kappa, \tag{2.3.3g}$$
$$\beta \to \beta + a\epsilon + \bar{a}\sigma + |a|^2\kappa, \tag{2.3.3h}$$
$$\gamma \to \gamma + a\alpha + \bar{a}(\beta + \tau) + |a|^2(\rho + \epsilon) + \bar{a}^2\sigma + \bar{a}|a|^2\kappa, \tag{2.3.3i}$$
$$\lambda \to \lambda + \bar{a}(2\alpha + \kappa) + \bar{a}^2(\rho + 2\epsilon) + \bar{a}^3\kappa + \bar{\delta}\bar{a} + \bar{a}D\bar{a}, \tag{2.3.3j}$$
$$\mu \to \mu + a\pi + 2\bar{a}\beta + 2|a|^2\epsilon + \bar{a}^2\sigma + a\bar{a}^2\kappa + \delta\bar{a} + aD\bar{a} \tag{2.3.3k}$$
$$\begin{aligned}\nu \to \nu &+ a\lambda + \bar{a}(\mu + 2\gamma) + \bar{a}^2(\tau + 2\beta) + \bar{a}^3\sigma + |a|^2(\pi + 2\alpha)\\ &+ a\bar{a}^2(\rho + 2\epsilon) + a\bar{a}^3\kappa + \left(\Delta + \bar{a}\delta + a\bar{\delta} + |a|^2 D\right)\bar{a},\end{aligned} \tag{2.3.3l}$$

and the corresponding changes in the Weyl scalar, are

$$\Psi_0 \to \Psi_0, \tag{2.3.4a}$$
$$\Psi_1 \to \Psi_1 + \bar{a}\Psi_1, \tag{2.3.4b}$$
$$\Psi_2 \to \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0, \tag{2.3.4c}$$
$$\Psi_3 \to \Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0, \tag{2.3.4d}$$
$$\Psi_4 \to \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0. \tag{2.3.4e}$$

Class II: Null rotations about $\boldsymbol{n}$

The null rotations around $\boldsymbol{n}$, with complex parameter, $b \in \mathbb{C}$, are defined in similar analogy to Sec. 2.3.1, but keeping $\boldsymbol{n}$ invariant, as

$$\boldsymbol{\ell} \to \boldsymbol{\ell} + \bar{b}\boldsymbol{m} + b\bar{\boldsymbol{m}} + |b|^2\boldsymbol{n}, \qquad \boldsymbol{n} \to \boldsymbol{n}, \qquad \boldsymbol{m} \to \boldsymbol{m} + b\boldsymbol{\ell}. \tag{2.3.5}$$

The effect of this transformation on the spin coefficients is obtained by accounting for the effect of the interchanging symmetry of $\boldsymbol{\ell}$ and $\boldsymbol{n}$, such that, $\boldsymbol{\ell} \leftrightarrow \boldsymbol{n}$ we have

$$\kappa \leftrightarrow -\bar{\nu}, \quad \rho \leftrightarrow -\bar{\mu}, \quad \sigma \leftrightarrow -\bar{\lambda}, \quad \alpha \leftrightarrow -\bar{\beta}, \quad \epsilon \leftrightarrow -\bar{\gamma}, \quad \pi \leftrightarrow -\bar{\tau}. \tag{2.3.6}$$

Thereby, the effect on the Weyl scalars are,

$$\Psi_0 \to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4, \tag{2.3.7a}$$

$$\Psi_1 \to \Psi_1 + +3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4, \tag{2.3.7b}$$
$$\Psi_2 \to \Psi_2 + 2b\Psi_3 + b^2\Psi_4, \tag{2.3.7c}$$
$$\Psi_3 \to \Psi_3 + b\Psi_4, \tag{2.3.7d}$$
$$\Psi_4 \to \Psi_4. \tag{2.3.7e}$$

which respect the interchange symmetry,

$$\Psi_0 \leftrightarrow \bar{\Psi}_4, \quad \Psi_1 \leftrightarrow \bar{\Psi}_3, \quad \Psi_2 \leftrightarrow \bar{\Psi}_2. \tag{2.3.8}$$

Class III: Boosts and Spin Rotations

To parameterize a boost in the null vectors normal to the hypersurface, $\boldsymbol{\ell}$ and $\boldsymbol{n}$, we characterise by a real scalar $A \in \mathbb{R}$, such that, the transformation is

$$\boldsymbol{\ell} \to A^{-1}\boldsymbol{\ell}, \qquad \boldsymbol{n} \to A\boldsymbol{n}, \qquad \boldsymbol{m} \to \boldsymbol{m}. \tag{2.3.9a}$$

Further, spin rotations correspond rotating the vectors $\boldsymbol{m}$, and $\bar{\boldsymbol{m}}$, within their plane, leaving $\boldsymbol{\ell}$ and $\boldsymbol{n}$ unchanged, parameterized by the rotation angle $\theta \in \mathbb{R}$, such that,

$$\boldsymbol{\ell} \to \boldsymbol{\ell}, \qquad \boldsymbol{n} \to \boldsymbol{n}, \qquad \boldsymbol{m} \to \exp(i\theta)\boldsymbol{m}. \tag{2.3.9b}$$

We further, combine both of these transformations, into the effective boost and spin rotation, characterised by the two real functions, as

$$\boldsymbol{\ell} \to A^{-1}\boldsymbol{\ell}, \qquad \boldsymbol{n} \to A\boldsymbol{n}, \qquad \boldsymbol{m} \to \exp(i\theta)\boldsymbol{m}. \tag{2.3.9c}$$

Under this transformation, we have the rotation coefficients, transforming as

$$\mu \to A\mu, \tag{2.3.10a}$$
$$\rho \to A^{-1}\rho, \tag{2.3.10b}$$
$$\tau \to \exp(i\theta)\tau, \tag{2.3.10c}$$
$$\pi \to \exp(-i\theta)\pi, \tag{2.3.10d}$$
$$\kappa \to A^{-2}\exp(i\theta)\kappa, \tag{2.3.10e}$$
$$\nu \to A^2\exp(-i\theta)\nu, \tag{2.3.10f}$$
$$\lambda \to A\exp(-2i\theta)\lambda, \tag{2.3.10g}$$
$$\sigma \to A^{-1}\exp(2i\theta)\sigma, \tag{2.3.10h}$$
$$\gamma \to A\gamma - \frac{1}{2}\Delta A + \frac{1}{2}iA\Delta\theta, \tag{2.3.10i}$$
$$\epsilon \to A^{-1}\epsilon - \frac{1}{2}A^{-2}DA + \frac{1}{2}iA^{-1}D\theta, \tag{2.3.10j}$$

$$\beta \to \exp(i\theta)\beta + \frac{1}{2}i\exp(i\theta)\delta\theta - \frac{1}{2}A^{-1}\exp(i\theta)\delta A, \tag{2.3.10k}$$

$$\alpha \to \exp(-i\theta)\alpha + \frac{1}{2}i\exp(-i\theta)\bar{\delta}\theta - \frac{1}{2}A^{-1}\exp(-i\theta)\bar{\delta}A. \tag{2.3.10l}$$

The transformed Weyl scalars are given as

$$\Psi_0 \to A^{-2}\exp(2i\theta)\Psi_0, \tag{2.3.11a}$$

$$\Psi_1 \to A^{-1}\exp(i\theta)\Psi_1, \tag{2.3.11b}$$

$$\Psi_2 \to \Psi_2, \tag{2.3.11c}$$

$$\Psi_3 \to A\exp(-i\theta)\Psi_3, \tag{2.3.11d}$$

$$\Psi_4 \to A^2\exp(-2i\theta)\Psi_4, \tag{2.3.11e}$$

where we can generalize the transformation as

$$\Psi_j \to A^{2-j}\exp\left((2-j)i\theta\right)\Psi_j. \tag{2.3.11f}$$

2.3.2 Petrov Classification

With respect to a given tetrad frame, the Weyl tensor is completely specified by the five complex scalars, $\Psi_0, \Psi_1, \ldots, \Psi_4$. These are scale dependent, the choice of frame, and on the six parameters of the group of Lorentz transformations. The Petrov classification is motivated by realising how many components of the Weyl tensor can be made to vanish by a suitable tetrad choice. The answer to this question gives rise to an algebraic classification of the Weyl tensor into types called the Petrov types. The Petrov classification describes the possible algebraic symmetries of the Weyl tensor.

Let $\Psi_4 \neq 0$[1]. Now, consider a rotation of Class II about $\boldsymbol{n}$, such that Ψ_4 remains invariant, we can set $\Psi_0 = 0$ by solving the quartic equation from Eq. (2.3.7), for $b \in \mathbb{C}$,

$$\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0. \tag{2.3.12}$$

This equation has always four roots, corresponding to new directions of $\boldsymbol{\ell}$, defined by $\boldsymbol{\ell} \to \boldsymbol{\ell} + \bar{b}\boldsymbol{m} + b\bar{\boldsymbol{m}} + |b|^2\boldsymbol{n}$, given by Eq. (2.3.5), which are termed the *principal null directions* of the Weyl tensor. If one or more roots coincide, the tensor is termed to be *algebraically special* (Types II, D, III, N, O); if no degeneracy, we have an *algebraically general* (Type I) spacetime. For Type O, the Weyl tensor vanishes, and we have a conformally flat spacetime.

The possible transitions between Petrov types are shown in the Fig. 2.1, which can also be interpreted as stating that some of the Petrov types are

[1] Even if it were to be zero in the chosen frame, then by a rotation of Class I about $\boldsymbol{\ell}$, leads to $\Psi_4 \to \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0$ from Eq. (2.3.4), which we can make non-zero, as long as not all the Weyl scalars vanish, or we have conformal flatness.

more special than others. The more algebraically special a spacetime is, the more components of the Weyl tensor can be made to vanish by suitably orienting the tetrad.

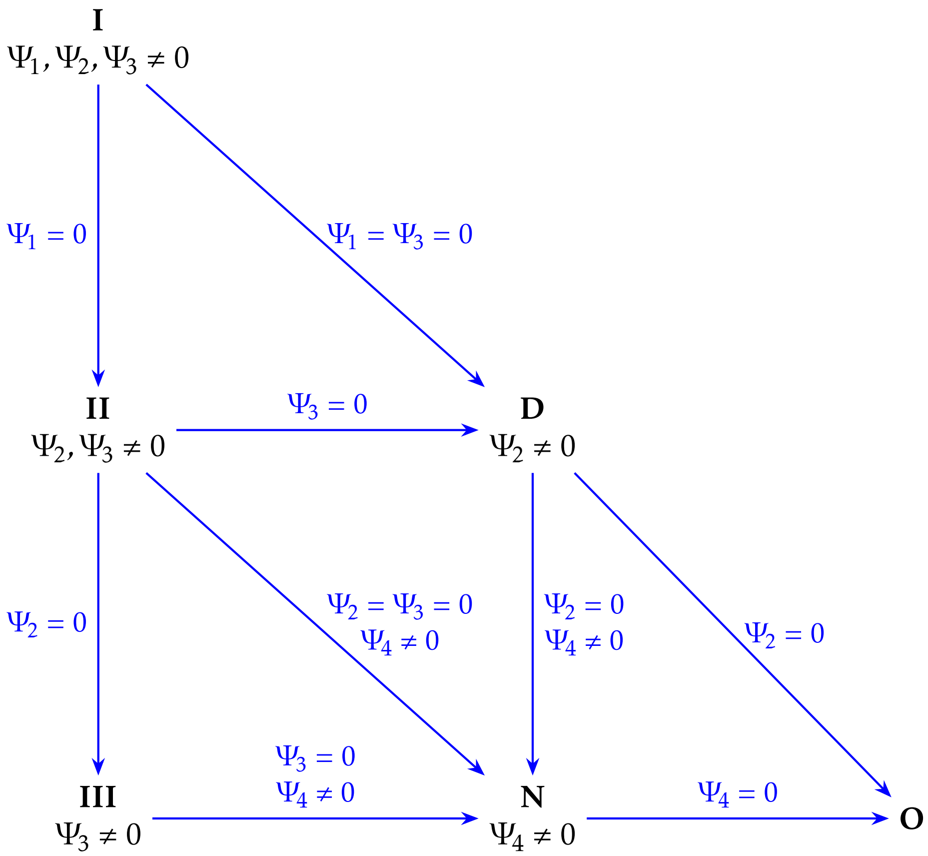

Figure 2.1: Penrose Representation of the Petrov classification. The Weyl scalars that cannot be eliminated by adjusting the tetrad in the direction of the principal null direction are shown below the corresponding spacetime type, relating to the number of degenerate roots of Eq. (2.3.12). The arrows indicate a progression towards more specialized solutions.

Type I Spacetime

In this case, Eq. (2.3.12) has four distinct roots, with which we ensure that $\Psi_0 = 0$ for each of the solutions, b, (generating a null rotation about $\boldsymbol{n}$). Choosing $b = b_\alpha$ which is one of the solution, we can now have the null rotation about $\boldsymbol{\ell}$ as a Class I transformation, that leaves Ψ_0 invariant, and we can set $\Psi_4 = 0$, by the relation in Eq. (2.3.4), for $b \in \mathbb{C}$,

$$\Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0 = 0, \tag{2.3.13}$$

whose roots, a, are the inverse of Eq. (2.3.12). Thus, with $\Psi_0 = \Psi_4 = 0$, we have Ψ_1, Ψ_2, Ψ_3 non-vanishing, of which $\Psi_1\Psi_3$ and Ψ_2 are invariant to the remaining rotations of Class III, being the boosts and spin transformations.

Type II Spacetime

For two coincident roots of Eq. (2.3.12), say, $b_\alpha = b_\beta$ the degeneracy helps us set $\Psi_1 = 0$, through its derivative,

$$\Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 = 0, \tag{2.3.14}$$

which is satisfied by the degenerate roots $b_\alpha = b_\beta$, and is the transformation equation of Class II for Ψ_1 in Eq. (2.3.14). Thereby, we can set $\Psi_0 = \Psi_1 = 0$ by a null rotation of $\boldsymbol{n}$ with the degenerate parameter. Similar to Type I spacetime, we can set $\Psi_4 = 0$ by a null rotation around $\boldsymbol{\ell}$ of Class I, which does not affect the simultaneous vanishing of Ψ_0 and Ψ_1. Only Ψ_2 and Ψ_3 are non-vanishing, and only Ψ_2 is invariant under rotations of Class III.

Type D Spacetime

Let Eq. (2.3.12) have two distinct degenerate roots b_α and b_γ, such that we have the transformation by a null rotation around $\boldsymbol{n}$ of Class II, rewritten,

$$\Psi_0 \to \Psi_4(b - b_\alpha)^2(b - b_\gamma)^2, \tag{2.3.15}$$

by explicitly factorizing.

Further, we obtain its derivatives, which lead us to the transformation equation of Class II for Ψ_1, Ψ_2, Ψ_3 in Eq. (2.3.14), to arrive at,

$$\Psi_1 \to \frac{1}{2}\Psi_4(b - b_\alpha)(b - b_\gamma)(2b - b_\alpha - b_\gamma), \tag{2.3.16a}$$

$$\Psi_2 \to \frac{1}{3}\Psi_4(b - b_\alpha)(b - b_\gamma) + \frac{1}{6}\Psi_4(2b - b_\alpha - b_\gamma)^2, \tag{2.3.16b}$$

$$\Psi_3 \to \frac{1}{2}\Psi_4(2b - b_\alpha - b_\gamma), \tag{2.3.16c}$$

and Ψ_4 remains invariant.

Thereby, we set $\Psi_0 = \Psi_1 = 0$ through $b = b_\alpha$, such that we have the simplification $\Psi_2 \to \frac{1}{6}\Psi_4(b_\alpha - b_\gamma)^2$ and $\Psi_3 \to \frac{1}{2}\Psi_4(b_\alpha - b_\gamma)$.

Now, we subject to a null rotation about $\boldsymbol{\ell}$ leading to a Class I transformation with parameter $\bar{a}$, where $\Psi_0 = \Psi_1 = 0$ are still simultaneously vanishing as per Eq. (2.3.4), and we have

$$\Psi_3 \to \frac{1}{2}\Psi_4(b_\alpha - b_\gamma)\Big[1 + \bar{a}(b_\alpha - b_\gamma)\Big], \tag{2.3.17a}$$

$$\Psi_4 \to \Psi_4\Big[1 + \bar{a}(b_\alpha - b_\gamma)\Big]^2, \tag{2.3.17b}$$

with $\Psi_2 \rightarrow \frac{1}{6}\Psi_4(b_\alpha - b_\gamma)^2$ remaining invariant.

Thus, we set $\bar{a} = \frac{1}{b_\alpha - b_\gamma}$, such that $\Psi_3 = \Psi_4 = 0$, through the rotation of Class I. Thereby, Ψ_2 is the only non-vanishing scalar, which is invariant to boosts and spin rotations of Class III.

Type III Spacetime

Supposing we have a multiplicity of three roots coinciding, $b_\alpha = b_\beta = b_\gamma$ for Eq. (2.3.12), then, by a null rotation about $\boldsymbol{n}$ of Class II, with parameter $b = b_\alpha$, we can make $\Psi_0 = \Psi_1 = \Psi_2 = 0$ simultaneously by the first (with $b = b_\alpha = b_\beta$), and second derivative (with $b = b_\alpha = b_\beta = b_\gamma$) leading to transformation relations.

Further, through a null rotation about $\boldsymbol{\ell}$ leading to the transformation of Class I, we can have $\Psi_4 = 0$, with the simultaneous vanishing of Ψ_0, Ψ_1, Ψ_2. The only non-vanishing scalar is Ψ_3, whose magnitude can be altered by boosts and spin transformations of Class III.

Type N Spacetime

When Eq. (2.3.12) is completely degenerate with four equal solutions, $b_\alpha = b_\beta = b_\gamma = b_\delta$, we can reduce $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ by a null rotation about $\boldsymbol{n}$ of Class II, with parameter $b = b_\alpha$, and its first (with $b = b_\alpha = b_\beta$), second (with $b = b_\alpha = b_\beta = b_\gamma$), and third derivative (with $b = b_\alpha = b_\beta = b_\gamma = b_\delta$) leading to transformation relations. Ψ_4 will remain non-vanishing, whose magnitude can be changed by rotations of Class III.

Goldberg-Sachs Theorem

Restricting to vacuum spacetimes where the Ricci tensor vanishes, and the Riemann and Weyl tensors coincide, we have the fundamental theorem highlighting the essence of the classification given by a reciprocal relationship between the spacetime type and the spin coefficients. Formally, if the Riemann tensor is of Type II and a null basis is so chosen that $\boldsymbol{\ell}$ is the repeated null direction and $\Psi_0 = \Psi_1 = 0$, then $\kappa = \sigma = 0$. Conversely, if $\kappa = \sigma = 0$, then $\Psi_0 = \Psi_1 = 0$ and the Riemann tensor is of Type II.

This characterization can be given a physical meaning as the existence of a congruence of shear-free null geodesics, as the integral curves of $\boldsymbol{\ell}$. A corollary to the Goldberg-Sachs theorem is that if the field is algebraically special and of Petrov type D, then the congruences formed by the two principal null-directions, $\boldsymbol{\ell}$ and $\boldsymbol{n}$, must both be geodesic and shear-free, i.e., $\kappa = \sigma = \nu = \lambda = 0$ when $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and conversely.

Remarkably, black hole solutions of General Relativity are all of Petrov Type D, and we can establish a null tetrad frame in which the spin coefficients $\kappa, \sigma, \nu, \lambda$ vanish trivially, and the only non-vanishing Weyl scalar is

Ψ_2. This highlights the effectiveness of the Newmann-Penrose formalism under consideration.

2.3.3 Spin Transformations

A related notion of spin rotation can lead to *spin weights*, which express the relative transformation of a tensor under a spin rotation of $\boldsymbol{m}$. Formally, for a tensor $\boldsymbol{T}$, projected onto the 2-surface spanned by $\{\boldsymbol{m}, \bar{\boldsymbol{m}}\}$, we have spin weight $\mathfrak{s}$, if the transformation of the tensor, is $\boldsymbol{T} \to \exp(i\mathfrak{s}\theta)\boldsymbol{T}$, when we have the spin rotation $\boldsymbol{m} \to \exp(i\theta)\boldsymbol{m}$.

As evident, the spin weights of $\boldsymbol{m}$ is $\mathfrak{s} = +1$, while $\bar{\boldsymbol{m}}$ has weight $\mathfrak{s} = -1$. From the previous section, we have the spin weights for the Ricci rotation coefficients, as μ, $\rho : \mathfrak{s} = 0$, which are invariant, τ, $\kappa : \mathfrak{s} = +1$, π, $\nu : \mathfrak{s} = -1$, which transform like $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ respectively, $\sigma : \mathfrak{s} = +2$, $\lambda : \mathfrak{s} = -2$, and γ, ϵ, β, α don't have a well-defined spin weight, and don't appear in covariant quantities. For the Weyl scalars, from Eq. (2.3.11), we have the spin weight, $\Psi_\nu : \mathfrak{s} = 2 - j$.

Consider a general tensor $\boldsymbol{T}$, of weight $\mathfrak{s} = \mathsf{p} - \mathsf{q}$, where, we have the projection tensor $\tilde{T} = m^{a_1} m^{a_2} \cdots m^{a_\mathsf{p}} \bar{m}^{b_1} \bar{m}^{b_2} \cdots \bar{m}^{b_\mathsf{q}} T_{a_1 \cdots b_\mathsf{q}}$, we further define the spin operators, $\eth$ as

$$\eth \tilde{T} = m^{a_1} m^{a_2} \cdots m^{a_\mathsf{p}} \bar{m}^{b_1} \bar{m}^{b_2} \cdots \bar{m}^{b_\mathsf{q}} \delta T_{a_1 \cdots b_\mathsf{q}}, \tag{2.3.18}$$

If we project of the angular operators on the 2-surface defined by $\mathrm{Span}\{\boldsymbol{m}, \bar{\boldsymbol{m}}\}$, we have

$$\eth \tilde{T} = \delta \tilde{T} + (\mathsf{p} - \mathsf{q})(\bar{\alpha} - \beta)\tilde{T} = \delta \tilde{T} + \mathfrak{s}(\bar{\alpha} - \beta)\tilde{T}, \tag{2.3.19}$$

where the effect of the spin operator $\eth$, is effectively raising the spin weight $\mathfrak{s}$. The conjugate $\bar{\eth}$ similarly lowers the spin weight.

Further, we can rewrite the field equations from Eq. (2.2.26) with the angular derivatives δ, $\bar{\delta}$, being written in terms of the projected spin operators $\eth$, $\bar{\eth}$, which are convenient for tensor projections, with the angular equations being simplified as,

$$\eth\rho - \bar{\eth}\sigma = \rho(\bar{\alpha} + \beta) - \sigma\left(\alpha + \bar{\beta}\right) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01}, \tag{2.3.20a}$$

$$\eth\alpha - \bar{\eth}\beta = (\mu\rho - \lambda\sigma) + |\alpha|^2 + |\beta|^2 - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda, \tag{2.3.20b}$$

$$\eth\lambda - \bar{\eth}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu\left(\alpha + \bar{\beta}\right) - \lambda(\bar{\alpha} + \beta) - \Psi_3 + \Phi_{21}, \tag{2.3.20c}$$

and the projection of the relevant evolution equations

$$D\rho - \bar{\eth}\kappa = \left(\rho^2 + |\sigma|^2\right) + (\epsilon + \bar{\epsilon})\rho - \kappa\left(3\alpha + \bar{\beta} - \pi\right) - \tau\bar{\kappa} + \Phi_{00}, \tag{2.3.20d}$$

$$D\sigma - \bar{\eth}\kappa = \sigma(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon}) - (\tau - \bar{\pi} + 3\beta + \bar{\alpha})\kappa + \Psi_0, \tag{2.3.20e}$$

$$D\lambda - \bar{\eth}\pi = (\rho\lambda + \sigma\lambda) + \left(\pi + \alpha - \bar{\beta}\right) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20}, \tag{2.3.20f}$$

$$D\mu - \bar{\eth}\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \beta + \bar{\alpha}) - \nu\kappa - \mu(\epsilon + \bar{\epsilon}) + \Psi_2 + 2\Lambda, \tag{2.3.20g}$$

where the remaining evolution equations and the radial field equations are independent of the angular derivatives.

2.4 Application to 2-surfaces

Consider a null geodesic congruence with tangent vector $\boldsymbol{\ell}$. As discussed in Section 1.2, the behaviour of this congruence is characterized by its expansion, shear, and twist on a transverse 2-dimensional spacelike cross-section $\mathscr{N}$. Eq. (2.2.13a) represents the deviation from the geodesic nature of $\boldsymbol{\ell}$. If $\kappa = 0$, we have $D\boldsymbol{\ell} = (\varepsilon + \bar{\varepsilon})\boldsymbol{\ell}$, thus representing an non-affinely parameterized geodesic. Further, if we have $\varepsilon + \bar{\varepsilon} = 0$, we note the affine characterization. Similarly, from Eq. (2.2.13e), we have the geodesic equation $\Delta\boldsymbol{n} = -(\gamma + \bar{\gamma})\boldsymbol{n}$, iff $\nu = 0$, and affine parameterization is recovered for $\gamma + \bar{\gamma} = 0$.

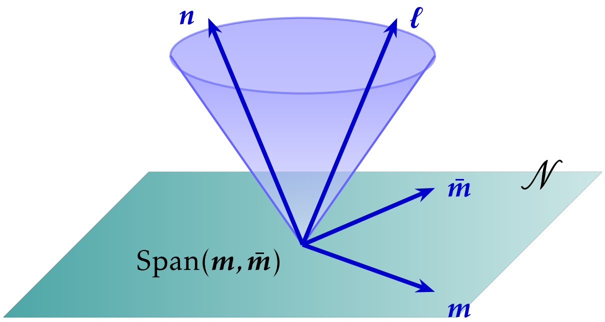

Figure 2.2: A diagram of a null basis at a point in the spacetime manifold. We have the construction of the null basis $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ adapted to the cross section $\mathscr{N}$ of the null hypersurface $\mathscr{H}$. The orthogonal complement of Span$(\boldsymbol{\ell}, \boldsymbol{n})$ is the transverse 2-surface, which is the $(\boldsymbol{m}, \bar{\boldsymbol{m}})$ plane.

The analysis of expansion, shear, and twist concerns the deformation of a 2-dimensional spacelike cross-section $\mathscr{N}$ orthogonal to the null vector $\boldsymbol{\ell}$. This 2-surface is spanned by the complex null vectors $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$. The induced metric $\boldsymbol{h}$ on this transverse 2-surface is given by the projection of the spacetime metric $\boldsymbol{g}$. For vectors $\boldsymbol{u}, \boldsymbol{v}$ restricted to this plane, the induced metric is

$$h_{\mu\nu} = m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu = 2m_{(\mu}\bar{m}_{\nu)}. \tag{2.4.1}$$

2.4.1 Intrinsic Curvature

For studying intrinsic curvature on the 2-surface, we can project the directional derivatives from Eq. (2.2.13i), (2.2.13j), on the $\{\boldsymbol{m}, \bar{\boldsymbol{m}}\}$ 2-surface, to obtain the following relations[2],

$$\delta \boldsymbol{m} \doteq (\beta - \bar{\alpha})\boldsymbol{m}, \tag{2.4.2a}$$

$$\bar{\delta} \boldsymbol{m} \doteq (\alpha - \bar{\beta})\boldsymbol{m}, \tag{2.4.2b}$$

These define the local curvature coefficients on the 2-surface defined by Span$(\boldsymbol{m}, \bar{\boldsymbol{m}})$.

We also note the commutation relation in Eq. (2.2.24d), which is

$$[\bar{\delta}, \delta] \doteq (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}, \tag{2.4.2c}$$

since we are interested in the connection locally, through a projection.

Let's compute the component ${}^{(2)}R_{m\bar{m}}$,

$$\begin{aligned}
{}^{(2)}R_{m\bar{m}} &\doteq {}^{(2)}R^{\alpha}_{mk\bar{m}} \doteq {}^{(2)}R^{m}_{mm\bar{m}} + {}^{(2)}R^{\bar{m}}_{m\bar{m}\bar{m}} \\
&\doteq \partial_m \Gamma^m_{m\bar{m}} - \partial_{\bar{m}} \Gamma^m_{mm} + \Gamma^p_{m\bar{m}}\Gamma^m_{pm} - \Gamma^p_{mm}\Gamma^m_{p\bar{m}} \\
&\doteq \delta(\alpha - \bar{\beta}) - \bar{\delta}(\beta - \bar{\alpha}) + 2(\alpha - \bar{\beta})(\beta - \bar{\alpha})
\end{aligned} \tag{2.4.3}$$

The Ricci scalar is the full contraction of the Ricci tensor with the inverse metric ${}^{(2)}R \doteq h^{\mu\nu}R_{\mu\nu} = -2R_{m\bar{m}}$. Thereby,

$$\begin{aligned}
{}^{(2)}R &\doteq -2\left[\delta(\alpha - \bar{\beta}) - \bar{\delta}(\beta - \bar{\alpha}) + 2(\alpha - \bar{\beta})(\beta - \bar{\alpha})\right] \\
&\doteq 4\left[(|\alpha|^2 + |\beta|^2) - \mathrm{Re}\{\delta\alpha - \bar{\delta}\beta\} - (\alpha\bar{\beta} + \bar{\alpha}\beta)\right]
\end{aligned} \tag{2.4.4}$$

Here we can further utilise the projections of the field equation and its complex conjugate in Eq. (2.2.26), such that, we have,

$${}^{(2)}R \doteq -4\,\mathrm{Re}\,\Psi_2 - 4\,\mathrm{Re}\{(\mu\rho - \lambda\sigma) + \Phi_{11} + \Lambda\} \tag{2.4.5}$$

Further, in two dimensions, any rank 2 antisymmetric tensor can be written as a pseudo scalar, such that $\mathcal{A}_{\mu\nu} \doteq \mathcal{A}\epsilon_{\mu\nu}$, such that $\mathcal{A}_{\mu\nu}\epsilon^{\mu\nu} \doteq 2\mathcal{A}$. We now proceed with deriving the Riemann tensor on the 2-surface, given by the antisymmetry, as

$$\begin{aligned}
{}^{(2)}R_{\mu\nu\rho\sigma} &\doteq \mathcal{K}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}, \\
&\doteq \mathcal{K}\left(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho}\right)
\end{aligned}$$

for some scalar function, $\mathcal{K}$, which is related to the scalar curvature. We can realise that, ${}^{(2)}R = 2\mathcal{K}$, by taking the trace of the Riemann tensor to arrive at the Ricci scalar, such that we have

$${}^{(2)}R_{\mu\nu\rho\sigma} \doteq \frac{1}{2}\,{}^{(2)}R\left(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho}\right) \doteq {}^{(2)}Rh_{\mu[\rho}h_{\sigma]\nu}. \tag{2.4.6}$$

[2] $\doteq$ stands for *equals, on the hypersurface, to.*

This further affirms that the number of independent components for the Riemann tensor in two dimensions is one, and is completely determined by the Ricci scalar. Expanding Eq. (2.4.6), with the metric, we have,

$$^{(2)}R_{\mu\nu\rho\sigma} \doteq \frac{1}{2}\,^{(2)}R(m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu)(m_\rho \bar{m}_\sigma - \bar{m}_\rho m_\sigma) \tag{2.4.7}$$

2.4.2 Extrinsic Quantities

The expansion $\theta_{(\boldsymbol{\ell})}$ is the trace of the deformation rate tensor, specifically the pull-back of $\nabla\boldsymbol{\ell}$ onto the transverse 2-surface. Substituting the inverse metric $h^{\mu\nu}$, we have

$$\begin{aligned}\theta_{(\boldsymbol{\ell})} &\doteq h^{\mu\nu}\nabla_\mu \ell_\nu \\ &\doteq m^\mu \bar{m}^\nu \nabla_\mu \ell_\nu + \bar{m}^\mu m^\nu \nabla_\mu \ell_\nu \\ &\doteq \bar{m}^\nu \delta \ell_\nu + m^\nu \bar{\delta} \ell_\nu.\end{aligned} \tag{2.4.8}$$

From the Newman-Penrose spin coefficient definitions, the spin coefficient ρ is defined as $\rho = m^\mu \bar{m}^\nu \nabla_\mu \ell_\nu = -\bar{m}^\nu \delta \ell_\nu$. as seen in Eq. (2.2.12h). Thus, the expansion simplifies as

$$\theta_{(\boldsymbol{\ell})} \doteq -\rho - \bar{\rho} = -2\,\mathrm{Re}\,\rho. \tag{2.4.9}$$

Further, we generalise the notion through the deformation rate $\boldsymbol{\Theta}$ of $\mathcal{N}$ along $\boldsymbol{\ell}$, by substituting the relevant rotation coefficients from Eq. (2.2.12h) and Eq. (2.2.12g), to arrive at,

$$\begin{aligned}\Theta_{\mu\nu} &\doteq h^\alpha_\mu h^\beta_\nu \nabla_\alpha \ell_\gamma, \\ &\doteq \left(m_\mu \bar{m}^\alpha + \bar{m}_\mu m^\alpha\right)\left(m_\nu \bar{m}^\beta + \bar{m}_\nu m^\beta\right)\nabla_\alpha \ell_\gamma \\ &\doteq -m_\mu m_\nu \bar{\sigma} - \bar{m}_\mu m_\nu \rho - m_\mu \bar{m}_\nu \bar{\rho} - \bar{m}_\mu \bar{m}_\nu \sigma,\end{aligned} \tag{2.4.10a}$$

with the trace-free symmetric part being the shear tensor of $\mathcal{N}$ along $\boldsymbol{\ell}$, simplified as,

$$\begin{aligned}\sigma_{\mu\nu} &\doteq \Theta_{(\mu\nu)} - \frac{1}{2}\theta_{(\boldsymbol{\ell})} h_{\mu\nu} \\ &\doteq -m_\mu m_\nu \bar{\sigma} - \bar{m}_\mu \bar{m}_\nu \sigma\end{aligned} \tag{2.4.10b}$$

and the antisymmetric component leading to the twist tensor as,

$$\begin{aligned}\tilde{\omega}_{\mu\nu} &\doteq \Theta_{[\mu\nu]} \\ &\doteq \left(\bar{m}_\mu m_\nu - m_\mu \bar{m}_\nu\right)(\bar{\rho} - \rho),\end{aligned} \tag{2.4.10c}$$

The twist scalar $\tilde{\omega}$ is defined by contracting with the area form $\varepsilon^{\mu\nu}$, as,

$$\tilde{\omega} \doteq \frac{1}{2}\varepsilon^{\mu\nu}\tilde{\omega}_{\mu\nu} \doteq \mathrm{Im}(\rho). \tag{2.4.10d}$$

where we have the antisymmetric area form $\varepsilon^{\mu\nu}$ defined through the outer product as,

$$\varepsilon^{\mu\nu} \doteq 2i\,(m^{\mu}m^{\nu} - m^{\nu}m^{\mu}) \tag{2.4.10e}$$

Thus, we see that $-2\,\mathrm{Re}\,\rho$ is the expansion, $\mathrm{Im}\,\rho$ is related to the twist scalar, and σ is the shear of $\boldsymbol{\ell}$. Similarly, the real and imaginary parts of μ give the expansion and twist of $\boldsymbol{n}$, while λ yields its shear.

Furthermore, we have the rotation 1-form defined in Eq, (1.2.14), which can be expanded in the dual basis, as

$$\begin{aligned}
\boldsymbol{\omega} &\doteq -\left(\boldsymbol{n}\nabla_{\Pi(\boldsymbol{\ell})}\boldsymbol{\ell}\right)\boldsymbol{n} + \left(\boldsymbol{n}\nabla_{\Pi(\boldsymbol{m})}\boldsymbol{\ell}\right)\bar{\boldsymbol{m}} + \left(\boldsymbol{n}\nabla_{\Pi(\bar{\boldsymbol{m}})}\boldsymbol{\ell}\right)\boldsymbol{m}\\
&\doteq -(\boldsymbol{n}\cdot D\boldsymbol{\ell})\,\boldsymbol{n} + (\boldsymbol{n}\cdot\delta\boldsymbol{\ell})\,\bar{\boldsymbol{m}} + \left(\boldsymbol{n}\cdot\bar{\delta}\boldsymbol{\ell}\right)\boldsymbol{m}\\
&\doteq -(\epsilon+\bar{\epsilon})\,\boldsymbol{n} + \left(\alpha+\bar{\beta}\right)\boldsymbol{m} + (\beta+\bar{\alpha})\,\bar{\boldsymbol{m}}
\end{aligned} \tag{2.4.11}$$

which are simplified using the relations between the directional derivatives in Eq. (2.2.13), and appropriate orthonormality conditions in Eq. (2.2.1). We will present an alternate notion in terms of the intrinsic derivative on the hypersurface for the 1-form, further.

An interesting quantity we encounter further is the curl of $\boldsymbol{\omega}$, which is contained in the exterior derivative $d\boldsymbol{\omega}$, which can be expressed in the tetrad formalism as,

$$d\boldsymbol{\omega} \doteq \left[\bar{\delta}\boldsymbol{\omega}(\bar{\delta}) - \delta\boldsymbol{\omega}(\delta) + \boldsymbol{\omega}\left([\delta,\bar{\delta}]\right)\right]\boldsymbol{m}\wedge\bar{\boldsymbol{m}}, \tag{2.4.12a}$$

where we have contracted the 1-form with the basis vectors on the hypersurface, and the Lie Bracket $[\delta,\bar{\delta}]$ is defined in Eq. (2.2.24d). The contracted 1-form can be evaluated as $\boldsymbol{\omega}(\delta) \doteq \beta + \bar{\alpha}$, and $\boldsymbol{\omega}(\bar{\delta}) \doteq \alpha + \bar{\beta}$. Expanding the Lie Bracket, we have $\boldsymbol{\omega}\left([\delta,\bar{\delta}]\right) \doteq -(\epsilon+\bar{\epsilon})(\mu-\bar{\mu}) + 2\left(\bar{\alpha}\bar{\beta} - \alpha\beta\right)$. Thereby, we have,

$$d\boldsymbol{\omega} \doteq \left[\left(\bar{\delta}\bar{\alpha} - \delta\bar{\beta}\right) - \left(\delta\alpha - \bar{\delta}\beta\right) - (\epsilon+\bar{\epsilon})(\mu-\bar{\mu}) + 2\left(\bar{\alpha}\bar{\beta} - \alpha\beta\right)\right]\boldsymbol{m}\wedge\bar{\boldsymbol{m}}, \tag{2.4.12b}$$

which bears a close resemblance to the imaginary component of the field equation projection in Eq. (2.2.26), such that we can simplify, as

$$d\boldsymbol{\omega} \doteq [2i\,\mathrm{Im}\,\Psi_2 - 2i\,\mathrm{Im}\{(\mu\rho - \lambda\sigma)\} - (\gamma+\bar{\gamma})(\rho-\bar{\rho}) - 2(\epsilon+\bar{\epsilon})(\mu-\bar{\mu})]\,\boldsymbol{m}\wedge\bar{\boldsymbol{m}}, \tag{2.4.12c}$$

Hence, the imaginary component of Ψ_2 comprises the curl of the rotation 1-form, while the real component characterizes the Gaussian curvature of the 2-surface.

2.4.3 Holomorphic Coordinates

We can simplify computations by working in a convenient set of holomorphic coordinates $(\xi,\bar{\xi})$. This is introduced by the transformation on the spin operators, $\eth$ acting on spin weight $\mathfrak{s} = 0$ scalars, $\tilde{T}$, as

$$\eth\tilde{T} \doteq P\frac{\partial}{\partial\xi}\tilde{T}, \tag{2.4.13a}$$

where $P = P(\xi, \bar{\xi})$ is a proportional factor, such that the action of the spin operator $\eth$, is a multiple of $\frac{\partial}{\partial\xi}$. We can further extend for general scalars $\tilde{T}$, with spin-weight $\mathfrak{s}$, as

$$\eth\tilde{T} \doteq P\bar{P}^{-\mathfrak{s}}\frac{\partial}{\partial\xi}\left(\bar{P}^{\mathfrak{s}}\tilde{T}\right), \tag{2.4.13b}$$

$$\bar{\eth}\tilde{T} \doteq \bar{P}P^{\mathfrak{s}}\frac{\partial}{\partial\bar{\xi}}\left(P^{-\mathfrak{s}}\tilde{T}\right), \tag{2.4.13c}$$

acting as effective raising and lowering operators for the spin weight. Using the above relations, we further have,

$$\left(\bar{\eth}\eth - \eth\bar{\eth}\right)\tilde{T} \doteq 2\mathfrak{s}\tilde{T}, \tag{2.4.13d}$$

as the commutation relations for the angular spin raising and lowering operators.

Thereby, we can read off the complex null 1-form $\boldsymbol{m}$ as

$$\boldsymbol{m} \doteq m_i dx^{\mu} \doteq \frac{1}{P}d\xi, \tag{2.4.14}$$

using which, the induced metric in Eq. (2.4.1) on the 2-surface, takes the form

$$ds^2 \doteq (\boldsymbol{m}\otimes\bar{\boldsymbol{m}} + \bar{\boldsymbol{m}}\otimes\boldsymbol{m}) \doteq \frac{2}{|P|^2}\,d\xi\;d\bar{\xi}. \tag{2.4.15}$$

Further, the exterior derivative $d\boldsymbol{m}$, is consequently derived from the above expressions, as

$$\begin{aligned} d\boldsymbol{m} &\doteq d\left(\frac{1}{P}\right)\wedge d\xi \\ &\doteq \left(-\frac{1}{P^2}\frac{\partial P}{\partial\bar{\xi}}\right)d\bar{\xi}\wedge d\xi \\ &\doteq -\frac{\partial P}{\partial\bar{\xi}}\bar{\boldsymbol{m}}\wedge\boldsymbol{m} \doteq \frac{\partial P}{\partial\bar{\xi}}\boldsymbol{m}\wedge\bar{\boldsymbol{m}}. \end{aligned} \tag{2.4.16}$$

We have the covariant derivative on the 2-surface encapsulated by $\alpha - \bar{\beta}$, as in Eq. (2.4.2), which can be related to the conformal factor, from the equations

$$\delta\boldsymbol{m} = m^{\mu}\nabla_{\mu}\boldsymbol{m} \doteq (\beta - \bar{\alpha})\boldsymbol{m}, \tag{2.4.17a}$$

where we now can simplify, using the expression for the exterior derivative in Eq. (2.4.16), to arrive at,

$$\beta - \bar{\alpha} \doteq \bar{m}^{\nu}m^{\mu}\nabla_{\mu}m_{\nu} \doteq \bar{m}^{\nu}m^{\mu}\left(\nabla_{\mu}m_{\nu} - \nabla_{\nu}m_{\mu}\right)$$

$$\doteq \bar{m}^{\nu} m^{\mu} (d\boldsymbol{m})_{\mu\nu} \doteq -\frac{\partial P}{\partial \bar{\xi}}, \tag{2.4.17b}$$

hence $\alpha - \bar{\beta} \doteq \frac{\partial \bar{P}}{\partial \xi}$.

Further, we have the Ricci scalar, ${}^{(2)}R$, for the 2-metric, in terms of P, as

$${}^{(2)}R \doteq 2|P|^2 \frac{\partial^2}{\partial \xi \partial \bar{\xi}} \ln |P|^2 \doteq \Delta_P \ln |P|^2, \tag{2.4.18}$$

where $\Delta_P \doteq 2|P|^2 \frac{\partial^2}{\partial \xi \partial \bar{\xi}}$ is the Laplacian of the 2-surface, termed as the *Laplace-Beltrami* operator.

Spherical Cross Sections

For 2-surfaces of spherical topology, parameterized by angular coordinates (θ, ϕ), we can write the action of the spin transformation in Eq. (2.3.18) on the projection of the general tensor $\boldsymbol{T}$, of weight $\mathfrak{s}$, as

$$\eth \tilde{T} \doteq -(\sin\theta)^{\mathfrak{s}} \left(\frac{\partial}{\partial \theta} + i \csc\theta \frac{\partial}{\partial \phi} \right) (\sin\theta)^{-\mathfrak{s}} \tilde{T} \tag{2.4.19a}$$

$$\bar{\eth} \tilde{T} \doteq -(\sin\theta)^{-\mathfrak{s}} \left(\frac{\partial}{\partial \theta} - i \csc\theta \frac{\partial}{\partial \phi} \right) (\sin\theta)^{\mathfrak{s}} \tilde{T}. \tag{2.4.19b}$$

where the spin operators transform equivalently to our earlier definition.

The holomorphic coordinates are obtained through stereographic projection, through the relation,

$$\xi \doteq \exp(i\phi) \cot\left(\frac{\theta}{2}\right). \tag{2.4.20}$$

such that we have $P \doteq \frac{1}{\sqrt{2}}\left(1 + \xi\bar{\xi}\right)$. The stereographic coordinate transformation for the sphere is constrained by the relations in Eq. (2.4.13).

Spin-Weighted Spherical Harmonics

From the notion of spherical harmonics on the surface of a sphere (parameterized by (θ, ϕ)), we extend to spin-$\mathfrak{s}$ spherical harmonics, ${}_{\mathfrak{s}}Y_{\ell,m}$ for integral values, as

$${}_{\mathfrak{s}}Y_{\ell,m}(\theta,\phi) = \begin{cases} \sqrt{\frac{(\ell-|\mathfrak{s}|)!}{(\ell+|\mathfrak{s}|)!}} (-1)^{\mathfrak{s}} \eth^{\mathfrak{s}} Y_{\ell,m}(\theta,\phi) & 0 \le \mathfrak{s} \le \ell \\ \sqrt{\frac{(\ell-|\mathfrak{s}|)!}{(\ell+|\mathfrak{s}|)!}} \bar{\eth}^{|\mathfrak{s}|} Y_{\ell,m}(\theta,\phi) & -\ell \le \mathfrak{s} \le 0 \end{cases}. \tag{2.4.21}$$

For each value of $\mathfrak{s}$, the ${}_{\mathfrak{s}}Y_{\ell,m}$ form a complete orthonormal set, such that we can expand any function, with spin-weight $\mathfrak{s}$. For negative spin-weight representation, we have,

$${}_{-\mathfrak{s}}\bar{Y}_{\ell,-m} = (-1)^{m+\mathfrak{s}}\, {}_{\mathfrak{s}}Y_{\ell,m} \tag{2.4.22}$$

We have some interesting properties, including

$$\eth\,{}_{\mathfrak{s}}Y_{\ell,m} = \sqrt{\frac{(\ell-\mathfrak{s})(\ell+\mathfrak{s}+1)}{2}}\,{}_{\mathfrak{s}+1}Y_{\ell,m}, \tag{2.4.23a}$$

$$\bar{\eth}\,{}_{\mathfrak{s}}Y_{\ell,m} = -\sqrt{\frac{(\ell+\mathfrak{s})(\ell-\mathfrak{s}+1)}{2}}\,{}_{\mathfrak{s}-1}Y_{\ell,m}, \tag{2.4.23b}$$

$$|\eth|^2\,{}_{\mathfrak{s}}Y_{\ell,m} = -\frac{(\ell-\mathfrak{s})(\ell+\mathfrak{s}+1)}{2}\,{}_{\mathfrak{s}}Y_{\ell,m}. \tag{2.4.23c}$$

Thereby, it is seen that ${}_{\mathfrak{s}}Y_{\ell,m}$ are the eigenfunctions of the operator $\bar{\eth}\eth$, which is related to the 2-surface gradient.

We can further rewrite the spin-weighted spherical harmonics defined in Eq. (2.4.21), in terms of the holomorphic coordinates as,

$$ {}_{\mathfrak{s}}Y_{\ell,m} = \frac{a_{\ell,m}}{\sqrt{(\ell-\mathfrak{s})!(\ell+\mathfrak{s})!}}\frac{1}{\left(1+\xi\bar{\xi}\right)^{\ell}}\sum_{p}\binom{\ell-\mathfrak{s}}{p}\binom{\ell+\mathfrak{s}}{p+\mathfrak{s}-m}\xi^{p}(-\bar{\xi})^{p+\mathfrak{s}-m}, \tag{2.4.24a}$$

where we have

$$a_{\ell,m} = (-1)^{\ell-m}\sqrt{\frac{1}{4\pi}(2\ell+1)(\ell+m)!(\ell-m)!}. \tag{2.4.24b}$$

with $\ell \geq 0$, $-\ell \leq m \leq \ell$, and $|\mathfrak{s}| \leq \ell$.

Part II

ISOLATED HORIZONS

The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.

SUBRAHMANYAN CHANDRASEKHAR

Chapter 3

Quasi-Local Horizons

3.1 Non Expanding Horizons

Naively, a *black hole* is a localised region of an n-dimensional spacetime $(\mathscr{M}, \boldsymbol{g})$ from which neither massive particles nor massless photons can escape. This characterises a surface through which no particle in the black hole region can cross, which is the *event horizon*. The event horizon has to be a *hypersurface* of the spacetime manifold $\mathscr{M}$, which exists as an embedded submanifold of codimension 1. The causal nature of the hypersurface is enforced by the nature of the induced metric onto the hypersurface, and equivalently by the nature of the normal vector to the surface. In advent of a quasi-local understanding of black holes, we proceed with defining the black hole horizon as a null hypersurface with relevant causal properties.

3.1.1 Trapped Surfaces

We begin with the notion of a *future trapped surface*, $\mathscr{N}$ as the compact cross-section, containing two systems of null geodesics, $\boldsymbol{\ell}$, $\boldsymbol{n}$, emerging orthogonally from $\mathscr{N}$, towards the future which converge locally, such that they have negative expansions

$$\theta_{(\boldsymbol{\ell})} < 0, \quad \theta_{(\boldsymbol{n})} < 0, \tag{3.1.1}$$

where we extend the definition of the expansion along $\boldsymbol{n}$ similarly as $\theta_{(\boldsymbol{n})} = h^{\mu\nu}\nabla_{\mu} n_{\nu}$. The expansion needs to be negative not only at a particular spacetime point, but over the whole cross section. This subtlety motivates the notion of quasi-locality, used further.

Further, a *marginally future trapped surface*, is the compact cross section $\mathscr{N}$ such that the expansion of the outgoing null vector $\boldsymbol{\ell}$, vanishes $\theta_{(\boldsymbol{\ell})} \doteq 0$ and the expansion of the ingoing null congruence $\boldsymbol{n}$, converges $\theta_{(\boldsymbol{n})} < 0$.

Thereby, a critical notion is of a *marginally outer trapped surface (MOTS)* which is a compact cross section $\mathscr{N}$, where the expansion of the future-directed, outward pointing normal $\boldsymbol{\ell}$, has vanishing expansion $\theta_{(\boldsymbol{\ell})} \doteq 0$, without restriction on $\boldsymbol{n}$. A surface with $\theta_{(\boldsymbol{\ell})} < 0$ and no restriction on $\boldsymbol{n}$, is called *outer trapped*.

To study black holes, we first analyse cases in equilibrium, by building the critical idea of a *non-expanding horizon (NEH)*, which can be defined as a null hypersurface $\mathscr{H}$, with topology $\mathscr{H} \simeq \mathbb{R} \times \mathscr{N}$, where $\mathscr{N}$ is a compact submanifold, such that the expansion of $\mathscr{H}$, along any null normal $\boldsymbol{\ell}$ vanishes identically. A direct consequence of the non-expanding nature results in an area $\mathscr{A}$ that does not depend on the choice of the complete cross section $\mathscr{N}$. We require that, all cross sections $\mathscr{N}$ of $\mathscr{H}$ have to be MOTS. We also require the additional condition of null energy condition, $T(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0$, for any future-directed null normal $\boldsymbol{\ell}$.

For $\mathscr{H}$ being an non-expanding horizon, the Raychaudhuri equation in Eq. (1.2.16) reduces to

$$\sigma_{\mu\nu}\sigma^{\mu\nu} + R(\boldsymbol{\ell}, \boldsymbol{\ell}) \doteq 0, \tag{3.1.2a}$$

with which we employ the null convergence relation $R(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0$, for any null vector $\boldsymbol{\ell}$, which is equivalent to the null energy condition, $T(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0$, which forces,

$$\sigma_{\mu\nu}\sigma^{\mu\nu} \doteq 0 \implies \boldsymbol{\sigma} \doteq 0, \quad R(\boldsymbol{\ell}, \boldsymbol{\ell}) \doteq 0. \tag{3.1.2b}$$

such that the deformation rate of any cross section $\mathscr{N}$ of a non-expanding horizon $\mathscr{H}$, along any null normal $\boldsymbol{\ell}$ is identically zero. This implies that the null direction tangent to $\mathscr{H}$ is covariantly constant on $\mathscr{H}$.

3.1.2 Induced Affine Connection

Since $\mathscr{H}$ is a null hypersurface, the metric induced is degenerate. As a consequence, there is a priori no unique connection on $\mathscr{H}$ associated with the induced metric $\boldsymbol{h}$.

However, when $\mathscr{H}$ is a non-expanding horizon and the null convergence condition holds on $\mathscr{H}$, so that $\theta_{(\boldsymbol{\ell})} \doteq 0$, then the spacetime connection ∇ induces a unique torsion-free connection $\hat{\nabla}$ on $\mathscr{H}$ as follows. Consider two vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$, and we realise that $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ is orthogonal to $\boldsymbol{\ell}$, since the deformation tensor is inherently the pull-back of the bilinear form $\nabla\underline{\boldsymbol{\ell}}$ $\boldsymbol{\Theta} \doteq \Phi^*\nabla\underline{\boldsymbol{\ell}}$ which implies that $\Phi^*\nabla\underline{\boldsymbol{\ell}} \doteq 0$. Thereby, an important consequence arising is that, for $u, v \in T_p\mathscr{H}$, we have

$$\boldsymbol{\ell} \cdot \nabla_{\boldsymbol{u}}\boldsymbol{v} \doteq \nabla_{\boldsymbol{u}}(\boldsymbol{\ell} \cdot \boldsymbol{v}) - \boldsymbol{v} \cdot \nabla_{\boldsymbol{u}}\boldsymbol{\ell} \doteq 0, \tag{3.1.3}$$

since $\boldsymbol{v} \cdot \nabla_{\boldsymbol{u}}\boldsymbol{\ell} \doteq \Phi^*\nabla\underline{\boldsymbol{\ell}}(\boldsymbol{u}, \boldsymbol{v}) \doteq 0$. Hence, the operator $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ is tangential to $\mathscr{H}$ and can describe the notion of an *induced affine connection* on $\mathscr{H}$.

We can formally define the induced affine connection on the null horizon $\mathscr{H}$, on the null horizon through the operator $\hat{\nabla}$ which projects the tuple $(\boldsymbol{u}, \boldsymbol{v})$ onto the tangent space of vector fields on the hypersurface $\mathscr{H}$, through the action $\nabla_{\boldsymbol{u}}\boldsymbol{v}$. Thereby, this invokes the natural affine connection induced on $\mathscr{H}$ by ∇. Hence, we have $\hat{\nabla}$, which projects the gradient onto the hypersurface, to yield the same for tangent vector fields to $\mathscr{H}$. In particular, when $\mathscr{H}$ is not a non-expanding horizon, we have

$$\hat{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \doteq \boldsymbol{\Pi}(\nabla_{\boldsymbol{u}}\boldsymbol{v}) \doteq \nabla_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\Theta}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{n}, \tag{3.1.4}$$

which is canonically independent of $\boldsymbol{n}$ for $\boldsymbol{\Theta} \doteq 0$, and becomes an intrinsic connection to $\mathscr{H}$. The compatibility of the intrinsic connection $\hat{\nabla}$ with the metric $\boldsymbol{h}$, is maintained with $\hat{\nabla}\boldsymbol{h} \doteq 0$. The action of the covariant connection can be extended uniquely to arbitrary tensor fields by the Lebinitz Rule.

Under the above condition, we revoke the covariant derivative of the null normal (as an analog of the extrinsic curvature) with respect to the affine connection $\hat{\nabla}$, which has the form,

$$\hat{\nabla}\boldsymbol{\ell} \doteq \boldsymbol{\ell} \otimes \boldsymbol{\omega}, \tag{3.1.5}$$

where $\boldsymbol{\omega}$ is the restriction of the 1-form defined in Eq. (1.2.14), as a tensor field on $\mathscr{H}$, to be seen as the pull-back of the inclusion map from $\mathscr{H}$ to $\mathscr{M}$. This can be seen through understanding of the $(1,1)$ tensor defined by $\hat{\nabla}\boldsymbol{\ell}$, which acts on a vector $\boldsymbol{u}$ and an 1-form $\boldsymbol{\Omega}$, as

$$\hat{\nabla}\boldsymbol{\ell}(\boldsymbol{\Omega}, \boldsymbol{v}) \doteq \boldsymbol{h}\left(\boldsymbol{\Omega}, \nabla_{\boldsymbol{u}}\boldsymbol{\ell}\right), \tag{3.1.6a}$$

and, in component form,

$$\left(\hat{\nabla}\boldsymbol{\ell}\right)^{\mu}_{\ \nu} \Omega_{\mu} v^{\nu} \doteq \Omega_{\mu} v^{\nu} \nabla_{\nu} \ell^{\mu}, \tag{3.1.6b}$$

which helps us understand the action of the 1-form on $\mathscr{M}$, using Eq. (1.2.13b)

$$\begin{aligned}
\hat{\nabla}\boldsymbol{\ell}(\boldsymbol{\Omega}, \boldsymbol{v}) &\doteq \left(\hat{\nabla}\boldsymbol{\ell}\right)^{\mu}_{\ \nu} \Omega_{\mu} v^{\nu} \doteq \Omega_{\mu} v^{\nu} \nabla_{\nu} \ell^{\mu} \\
&\doteq \Omega_{\mu} v^{\nu} \left(\Theta^{\mu}_{\nu} + \omega_{\nu} \ell^{\mu} - \ell_{\nu} n^{\gamma} \nabla_{\gamma} \ell_{\mu}\right) \\
&\doteq \Omega_{\mu} v^{\nu} \omega_{\nu} \ell^{\mu} - \Omega_{\mu} v^{\nu} \ell_{\nu} n^{\gamma} \nabla_{\gamma} \ell_{\mu} \\
&\doteq \Omega_{\mu} \ell^{\mu} v^{\nu} \omega_{\nu} \doteq \boldsymbol{h}(\Omega, \boldsymbol{\ell})\, \boldsymbol{h}(\omega, \boldsymbol{v}),
\end{aligned} \tag{3.1.6c}$$

since the deformation rate vanishes on the hypersurface $\mathscr{H}$, and $\ell_{\nu} v^{\nu} \doteq 0$. We thereby have,

$$\hat{\nabla}\boldsymbol{\ell}(\boldsymbol{\Omega}, \boldsymbol{v}) \doteq \boldsymbol{h}(\boldsymbol{\Omega}, \boldsymbol{\ell})\, \boldsymbol{h}(\omega, \boldsymbol{v}), \tag{3.1.6d}$$

thereby, proving the above claim, $\hat{\nabla}\boldsymbol{\ell} \doteq \boldsymbol{\ell} \otimes \boldsymbol{\omega}$.

Although, the choice of the 1-form $\boldsymbol{\omega}$ usually depends on the choice of the cross-section $\mathscr{N}$, for non-expanding horizons, we note that $\boldsymbol{\omega}$ is intrinsic to $\mathscr{H}$ and the choice of $\boldsymbol{\ell}$, and is usually termed the connection 1-form or rotation 1-form of $(\mathscr{H}, \boldsymbol{\ell})$.

3.2 Isolated Horizons

3.2.1 Weakly Isolated Horizons

From the previous subsection, we have introduced the notion of the NEH, which captures some fundamental and rich features of a black hole in quasi-equilibrium. As noted, we can have the notion of the induced derivative operator $\hat{\mathbf{\nabla}}$ on a NEH $\mathscr{H}$, such that, we specify the pair $(\boldsymbol{h}, \hat{\mathbf{\nabla}})$, that characterizes the geometry of $\mathscr{H}$.

As realised earlier, the NEH notion is independent of the rescaling of the null normal $\boldsymbol{\ell}$. However, imposing the constancy of the rotation 1-form under the flow of null normal $\boldsymbol{\ell}$, does depend on the choice of $\boldsymbol{\ell}$, since we have the concrete scaling relation, $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell}$, leads to $\boldsymbol{\omega}' = \boldsymbol{\omega} + \mathbf{\Pi}^* d \ln \alpha$, as seen in Sec. 1.2.3. Now, if we consider the null normal $\boldsymbol{\ell}$, such that we impose constancy of $\boldsymbol{\omega}$ under the flow of $\boldsymbol{\ell}$, expressed by the intrinsic Lie derivative on the horizon, as $\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\omega} \doteq 0$, then, after a rescaling of $\boldsymbol{\ell}$, the new rotation 1-form may not be time-independent, in general. To perform this, we resort to the invariance of $\boldsymbol{\omega}$, through defining a scale of equivalence in the null normals. Due to the non-existence of a canonical scaling to the null generators of the horizon, we define an equivalence class of null normals $[\boldsymbol{\ell}]$, such that $\boldsymbol{\ell}$ and $\tilde{\boldsymbol{\ell}}$ belong to the same class, if $\tilde{\boldsymbol{\ell}} = c\boldsymbol{\ell}$ for some positive constant c, which helps us make an explicit choice of the null normal.

Thereby, the pair $(\mathscr{H}, [\boldsymbol{\ell}])$ comprises a *weakly isolated horizon (WIH)*, if $\mathscr{H}$ is a NEH, and for each null normal $\boldsymbol{\ell}$ in the equivalence class $[\boldsymbol{\ell}]$, we have

$$\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\omega} \doteq 0. \tag{3.2.1}$$

While on a NEH, we have the time-independence of the intrinsic metric $\boldsymbol{h}$, the analog of the extrinsic curvature $\hat{\nabla}\boldsymbol{\ell}$ given by $\boldsymbol{K} = \hat{\nabla}\boldsymbol{\ell} = \boldsymbol{\ell} \otimes \boldsymbol{\omega}$, is also required to be time-independent on a WIH. However, the full connection $\hat{\nabla}$, nor curvature components, can still be time-dependent on a WIH.

Further, we realise that the machinery to prove the zeroth law of black hole mechanics through the expansion of the Lie derivative on the hypersurface, as

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\omega} &\doteq \boldsymbol{\ell} \cdot d\boldsymbol{\omega} + d\boldsymbol{\omega}(\boldsymbol{\ell})\Big|_{\mathscr{H}} \\ &\doteq \boldsymbol{\ell} \cdot d\boldsymbol{\omega} + \hat{\nabla}\kappa_{(\boldsymbol{\ell})} \doteq 0, \end{aligned} \tag{3.2.2}$$

to arrive at $\hat{\nabla}\kappa_{(\boldsymbol{\ell})} \doteq 0$, since the exterior derivative is restricted to the null hypersurface. Thus, the zeroth law of black hole mechanics naturally extends to WIHs.

Since the WIH is equipped with an equivalence class of null normals $[\boldsymbol{\ell}]$ under constant scaling, the surface gravity $\kappa_{(\boldsymbol{\ell})}$ does not have a canonical value on WIHs, unless it vanishes, since it scales with the null normal. Thus, we have a natural distinction between *extremal* WIHs, where $\kappa_{(\boldsymbol{\ell})} = 0$, and *non-extremal* WIHs where it does not vanish.

Extremal WIHs

Given a NEH $\mathscr{H}$, we can choose a null normal class $[\boldsymbol{\ell}]$, such that $(\mathscr{H}, [\boldsymbol{\ell}])$ is an extremal WIH, such that the integral curves of $\boldsymbol{\ell} \in [\boldsymbol{\ell}]$ are affinely parameterized geodesics. We realise that every NEH admits a family of null normals $\boldsymbol{\ell}, \boldsymbol{\ell}', \ldots$, which are related through a functional scaling, such that $(\mathscr{H}, [\boldsymbol{\ell}]), (\mathscr{H}, [\boldsymbol{\ell}']), \ldots$ are all extremal WIHs, with

$$\boldsymbol{\ell}' \doteq \frac{1}{f}\boldsymbol{\ell}, \quad \mathcal{L}_{\boldsymbol{\ell}} f \doteq 0, \tag{3.2.3}$$

making f an angular function thereby.

There exists an infinite degree of freedom in the construction of a WIH for the extremal case, due to the arbitrary function involved in the scaling. Given a NEH $\mathscr{H}$, we can choose an equivalence class $[\boldsymbol{\ell}]$, by restricting the null normals appropriately, such that we have an extremal WIH $(\mathscr{H}, [\boldsymbol{\ell}])$.

Non-Extremal WIHs

Given the class of null normals $[\boldsymbol{\ell}]$ that describe a fiducial extremal WIH, we can map any null normal $\boldsymbol{\ell} \in [\boldsymbol{\ell}]$ to a non-extrmal WIH $(\mathscr{H}, [\boldsymbol{\ell}'])$, where $\boldsymbol{\ell}' \in [\boldsymbol{\ell}']$ with surface gravity $\kappa_{(\boldsymbol{\ell}')}$, through

$$\boldsymbol{\ell}' \doteq \kappa_{(\boldsymbol{\ell}')}(\mathsf{v} - g)\boldsymbol{\ell}, \quad \mathcal{L}_{\boldsymbol{\ell}} g \doteq 0, \tag{3.2.4}$$

where v is a *compatible* coordinate, with $\mathcal{L}_{\boldsymbol{\ell}}\mathsf{v} = 1$. For v' corresponding to $\boldsymbol{\ell}'$ with $\mathcal{L}_{\boldsymbol{\ell}'}\mathsf{v}' = 1$, we have,

$$\mathsf{v}' = \frac{1}{\kappa_{(\boldsymbol{\ell}')}} \ln\Big(\kappa_{(\boldsymbol{\ell}')}(\mathsf{v} - g)\Big). \tag{3.2.5}$$

Thereby, we can restrict the class of null normals $\boldsymbol{\ell}'$ on a NEH $\mathscr{H}$ to lie on a suitable equivalence class $[\boldsymbol{\ell}']$ to arrive at an non-extremal WIH, $(\mathscr{H}, [\boldsymbol{\ell}'])$. As previously mentioned, there exists an infinite-dimensional freedom in this construction, due to the arbitrary function involved. We could also have begun with a non-extremal WIH, and construct an extremal WIH through the inverse of the above transformation for the null normal.

3.2.2 Isolated Horizons

We realise the intrinsic nature of the affine connection, given by Eq. (3.1.5), we have $\hat{\nabla}\boldsymbol{\ell} = \boldsymbol{\ell} \otimes \boldsymbol{\omega}$, which can be used to note,

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\ell}}\Big(\hat{\nabla}\boldsymbol{\ell}\Big) &\doteq \mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\ell} \otimes \boldsymbol{\omega} + \boldsymbol{\ell} \otimes \mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\omega} \\ &\doteq (\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\ell}) \otimes \boldsymbol{\omega} \doteq \hat{\nabla}(\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\ell}), \end{aligned} \tag{3.2.6a}$$

hence we have retrieve the identity,

$$\Big[\mathcal{L}_{\boldsymbol{\ell}}, \hat{\nabla}\Big]\boldsymbol{\ell} \doteq 0. \tag{3.2.6b}$$

This formulation implies a restriction on the intrinsic nature of the horizon, described by the induced affine connection, $\hat{\nabla}$. This notion of independence can be extended to all components to arrive at the general formulation of isolated horizons.

$(\mathscr{H}, [\boldsymbol{\ell}])$ constitutes an *Isolated Horizon (IH)*, given $\mathscr{H}$ is a NEH, and each null normal $\boldsymbol{\ell}$ in $[\boldsymbol{\ell}]$ satisfies,

$$\left[\mathcal{L}_{\boldsymbol{\ell}}, \hat{\nabla}\right] \doteq 0. \tag{3.2.7}$$

The intrinsic geometry of the IH is more restrictive than that of a WIH, and strengthens the notion of isolation by requiring the intrinsic metric $\boldsymbol{h}$ and the full derivative operator $\hat{\nabla}$ to be time independent. For generic NEHs, we can obtain the canonical choice of the null normal $\boldsymbol{\ell}$ through the solution of an invertible elliptic differential equation, due to a non-trivial kernel, which is not the case for IHs.

As discussed previously, we have the notion of extremal IHs if $\kappa_{(\boldsymbol{\ell})} = 0$ for $\boldsymbol{\ell} \in [\boldsymbol{\ell}]$, and non-extremal IHs for non-vanishing surface gravity. However, due to the nature of the restriction above in Eq. (3.2.7), we can't primarily construct an IH from every NEH. As we transition from a NEH to an IH, we assume that additional intrinsic geometrical structures remain time-independent. In the case of a NEH, the intrinsic metric $\boldsymbol{h}$ is time-independent. For a WIH, the analogous extrinsic curvature is also time-independent. Finally, for an IH, the entire intrinsic geometry is considered to be time-independent.

3.2.3 Mass, Angular Momentum, and Multipoles

For a WIH $(\mathscr{H}, [\boldsymbol{\ell}])$, we have $\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\omega} \doteq 0$ by definition, $\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\ell} \doteq \kappa_{(\boldsymbol{\ell})}\boldsymbol{\ell}$ with $\hat{\nabla}\kappa_{(\boldsymbol{\ell})} = 0$ from the geodesic nature of $\boldsymbol{\ell}$, and $\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{h} \doteq 0$ by constancy of the intrinsic metric along null generators of $\boldsymbol{\ell}$, where $\boldsymbol{\ell} \in [\boldsymbol{\ell}]$. We realise $\boldsymbol{\ell}$ as a *symmetry* of the horizon (or infinitesimal automorphism) since its flow preserves both the intrinsic metric and the intrinsic connection.

We formally define a vector field $\boldsymbol{\varphi}$ tangent to a WIH $(\mathscr{H}, [\boldsymbol{\ell}])$ as a symmetry of the horizon, if it satisfies,

$$\mathcal{L}_{\boldsymbol{\varphi}}\boldsymbol{\ell} \doteq c_{\boldsymbol{\varphi}}\boldsymbol{\ell}, \tag{3.2.8a}$$

$$\mathcal{L}_{\boldsymbol{\varphi}}\boldsymbol{h} \doteq 0, \tag{3.2.8b}$$

$$\mathcal{L}_{\boldsymbol{\varphi}}\boldsymbol{\omega} \doteq 0, \tag{3.2.8c}$$

where $c_{\boldsymbol{\varphi}}$ is constant on $\mathscr{H}$ with $\hat{\nabla}c_{\boldsymbol{\varphi}} = 0$. This means that $\boldsymbol{\varphi}$ must preserve the equivalence class $[\boldsymbol{\ell}]$ of null normals that are prescribed for the WIH, the intrinsic metric, and the rotation 1-form.

The set of vector fields $\boldsymbol{\varphi}$ satisfying the above constraints is the infinitesimal symmetries of the horizon, and forms a symmetry Lie algebra $\mathcal{S}$ under the usual commutator bracket. Irrespective of the characteristics of a WIH, $\mathcal{S}$ is at least one-dimensional, since $\boldsymbol{\ell} \in \mathcal{S}$ as noted earlier. These infinitesimal symmetries preserve each integral curve of $\boldsymbol{\ell}$.

The set of all WIHs can be divided into *Spherically Symmetric*, *Axi-symmetric*, and *Generic* based on the dimension of the symmetry group $\mathcal{S}$. For spherically symmetric WIHs, the horizon symmetry algebra is four-dimensional, while axi-symmetric WIHs have a two-dimensional Abelian Lie algebra, and generic WIHs have a one-dimensional Lie algebra.

Axial Symmetry

For axi-symmetric spacetimes, we have the horizon symmetry generators $\{\boldsymbol{\ell}, \boldsymbol{\phi}\}$, where $\boldsymbol{\phi}$ is a rotational vector field having closed integral curves, an affine length of 2π, and should vanish at exactly two null horizon generators. We can define the horizon angular momentum quasi-locally, as the surface term at $\mathscr{H}$ in the Hamiltonian, which generates diffeomorphisms along a rotational vector field, which is equal to $\boldsymbol{\phi}$ at the horizon, through

$$\mathsf{J} \doteq -\frac{1}{8\pi} \oint_{\mathscr{N}} (\boldsymbol{\phi} \cdot \boldsymbol{\omega})\, \boldsymbol{\epsilon}, \tag{3.2.9}$$

where $\mathscr{N}$ is the cross section and J is a geometric invariant independent of the choice of $\mathscr{N}$.

Since Eq. (2.4.12) relates the curl of the exterior 1-form $d\boldsymbol{\omega}$ with the imaginary part of Ψ_2, which we shall realise as a geometric invariant on $\mathscr{H}$, we can rewrite the angular momentum, as

$$\mathsf{J} \doteq -\frac{1}{4\pi} \oint_{\mathscr{N}} \zeta \,\mathrm{Im}\, \Psi_2\, \boldsymbol{\epsilon}, \tag{3.2.10a}$$

where ζ is an unique globally defined function, as a *potential* for $\boldsymbol{\phi}$,

$$\hat{\nabla}_\nu \zeta \doteq \frac{1}{\mathsf{R}^2} \phi^\mu \epsilon_{\mu\nu}, \qquad \oint_{\mathscr{N}} \zeta\, \boldsymbol{\epsilon} \doteq 0. \tag{3.2.10b}$$

where R is the area radius of the cross-section, with area $4\pi\mathsf{R}^2$. ζ is completely determined by the metric $\boldsymbol{h}$ and the rotational field $\boldsymbol{\phi}$.

For the notion of energy, we relate to the time translations through the time evolution vector $\boldsymbol{t} \doteq \mathsf{A}\boldsymbol{\ell} + \Omega\boldsymbol{\phi}$, where A, Ω are constant on a given WIH but vary over phase space. Hamiltonian considerations lead to an expression for the mass as

$$\mathsf{M} \doteq \frac{1}{2\mathsf{R}} \sqrt{\mathsf{R}^4 + 4\mathsf{J}^2}. \tag{3.2.11}$$

For a non-spinning black hole, this reduces to the Schwarzschild expression $\mathsf{M} \doteq \frac{1}{2}\mathsf{R}$. The Hamiltonian analysis further relates the surface gravity for the integral curves of $\boldsymbol{\ell}$, as

$$\kappa_{(\boldsymbol{\ell})} \doteq \frac{\mathsf{R}^4 - 4\mathsf{J}^2}{2\mathsf{R}^3 \sqrt{\mathsf{R}^4 + 4\mathsf{J}^2}}, \tag{3.2.12}$$

which is the usual expression for surface gravity for a Kerr metric, and reduces to the Schwarzschild expression, $\kappa_{(\boldsymbol{\ell})} \doteq \frac{1}{2\mathsf{R}} \doteq \frac{1}{4\mathsf{M}}$.

Further, we expand Ψ_2 in terms of the set of invariant coordinates ζ and orthonormal spherical harmonics $Y_\ell^m(\zeta)$ for axi-symmetric spacetimes, leading to the *multipole expansion*,

$$\mathsf{I}_\ell + i\mathsf{L}_\ell \doteq -\oint_{\mathcal{N}} \Psi_2 Y_\ell^0(\zeta)\, \boldsymbol{\epsilon} \tag{3.2.13}$$

where the normalization for the spherical harmonics is

$$\oint_{\mathcal{N}} Y_\ell^0(\zeta) Y_{\ell'}^0(\zeta)\, \boldsymbol{\epsilon} \doteq \mathsf{R}^2 \delta_{\ell,\ell'}. \tag{3.2.14}$$

The multipoles are diffeomorphism invariant due to the dependency of ζ on the metric and the rotational field. I_ℓ and L_ℓ are not freely specified entirely freely but are subject to certain algebraic constraints. The Gauss-Bonet theorem constraints $\mathsf{I}_0 = \sqrt{\pi}$ due to the real part of Ψ_2 leading to the scalar curvature of the cross section $\mathcal{N}$. We further have $\mathsf{L}_0 = 0$ and L_1 being proportional to the angular momentum, since $\operatorname{Im}\Psi_2$ is related to the curl of the exterior 1-form $\boldsymbol{\omega}$.

The dimensionless multipoles can be rescaled into mass and angular momentum multipoles, which provide a diffeomorphism invariant characterization of the horizon geometry. The knowledge of the area and multipole moments suffices to reconstruct an isolated horizon geometry.

Electromagnetism

The notion of angular momentum can be extended to axial symmetries on the horizon, which preserves the metric, the rotation 1-form, and the electromagnetic field on the horizon,

$$\mathsf{J} \doteq -\frac{1}{4\pi}\oint_{\mathcal{N}} \zeta \operatorname{Im}\Psi_2\, \boldsymbol{\epsilon} + \frac{1}{2\pi}\oint_{\mathcal{N}} \xi \operatorname{Im}\phi_1\, \boldsymbol{\epsilon}, \tag{3.2.15}$$

where ξ is the *potential* for the Maxwell's field, defined by,

$$\hat{\nabla}_\mu \xi = \frac{1}{2\mathsf{R}^2}\phi_\mu \epsilon^{\gamma\delta} F_{\gamma\delta}. \tag{3.2.16}$$

Further, we define the notion of electric and magnetic charges of the horizon,

$$\mathsf{Q} + i\mathsf{P} \doteq \frac{1}{2\pi}\oint_{\mathcal{N}} \phi_1\, \boldsymbol{\epsilon}. \tag{3.2.17}$$

As previously noted, Hamiltonian methods provide a suitable notion of horizon mass, and defining $\mathsf{q} = \sqrt{\mathsf{Q}^2 + \mathsf{P}^2}$,

$$\mathsf{M} \doteq \frac{1}{2\mathsf{R}}\sqrt{(\mathsf{R}^2 + \mathsf{q}^2)^2 + 4\mathsf{J}^2}, \tag{3.2.18}$$

and the surface gravity,

$$\kappa_{(\ell)} \doteq \frac{\mathsf{R}^4 - \mathsf{q}^4 + 4\mathsf{J}^2}{2\mathsf{R}^3\sqrt{(\mathsf{R}^2 + \mathsf{q}^2)^2 + 4\mathsf{J}^2}}, \tag{3.2.19}$$

which reduces to Eq. (3.2.11) and Eq. (3.2.12) respectively in the lack of electromagnetic fields.

3.3 Geometry of Horizons

We analyze in detail the geometrical structures intrinsically defined on WIHs and IHs, primarily null normals $\boldsymbol{\ell}$, the intrinsic (degenerate) metric $\boldsymbol{h}$, and the intrinsic derivative operator $\hat{\nabla}$, and study their relationship with Einstein's equations, potentially including matter sources. By studying their relationships, we derive the constraints they must satisfy due to the pull-back of the field equations to horizons, while specifying the free data.

3.3.1 Constraint Equations

To freely specify data on WIHs, we require $(\boldsymbol{h}, [\boldsymbol{\ell}], \hat{\nabla})$ to follow specific constraints. We introduce the transversal tensor defining the intrinsic variation of the transverse null normal,

$$\boldsymbol{S} \doteq \hat{\nabla}\boldsymbol{n}, \tag{3.3.1}$$

which, without loss of generality, can be assumed to be symmetric. Due to the normalization, $\boldsymbol{\ell} \cdot \boldsymbol{n} = -1$, we have $\boldsymbol{\omega}(\boldsymbol{v}) \doteq \boldsymbol{S}(\boldsymbol{\ell}, \boldsymbol{v})$ for $\boldsymbol{v} \in T_p\mathscr{M}$. The trace and trace-free parts of $\boldsymbol{S}$ yield the expansion and shear of $\boldsymbol{n}$. The complete characterization of $\mathscr{H}$ requires a specification of $\boldsymbol{S}$ everywhere on $\mathscr{H}$.

For a WIH, using Einstein's equations in Eq. (1.1.7), we can show that the components of ${}^{(2)}\boldsymbol{R}$ transverse to $\mathscr{H}$ dictate the evolution, given by,

$$\mathcal{L}_{\boldsymbol{\ell}} S_{\mu\nu} \doteq -\kappa_{(\boldsymbol{\ell})} S_{\mu\nu} + \hat{\nabla}_{(\mu}\omega_{\nu)} + \omega_\mu\omega_\nu - \frac{1}{2}{}^{(2)}R_{\mu\nu} + T_{\mu\nu} - \frac{1}{2}T h_{\mu\nu}, \tag{3.3.2}$$

where the equality holds on the cross-section. By specifying $\boldsymbol{S}$ on some initial cross-section, a solution of this constraint equation then yields $\boldsymbol{S}$ everywhere on $\mathscr{H}$. The time derivative of $\boldsymbol{S}$ yields the constraint equations to be satisfied by the initial data, along with the zeroth law, asserting the constancy of $\kappa_{(\boldsymbol{\ell})}$.

In the standard 3+1 decomposition in Sec. 1.3.1, the pullback of the space-time Ricci tensor to the submanifolds yields the evolution equations along the normal to the surface. In the present case, since the normal is also tangential, the analogous equation is now a constraint. These geometric quantities and identities can be employed to choose a suitable equivalence class $[\boldsymbol{\ell}]$ on a NEH and thereby find an admissible IH.

Given the pair $(\mathscr{H}, [\boldsymbol{\ell}])$ with closed cross sections, we can construct a WIH with metric $\boldsymbol{h}$ and intrinsic affine connection $\hat{\nabla}$, by choosing a cross-section and fixing the 2-metric, and carry the fields on the horizon by following the requisite constraints. Thus, given $(\mathscr{H}, [\boldsymbol{\ell}])$, the free data consists of a constant $\kappa_{(\boldsymbol{\ell})}$ and fields $\boldsymbol{q}, \boldsymbol{\omega}, \boldsymbol{S}$ on any 2-sphere cross-section of $\mathscr{H}$. In the Newman-Penrose notation, $\boldsymbol{\omega}$ is specified by the spin coefficient π and $\boldsymbol{S}$ is determined by the transversal coefficients μ and λ.

3.3.2 Canonical Foliation

A WIH has generators tangent to the horizon and a rotation 1-form $\boldsymbol{\omega}$, intrinsically defined by the intrinsic connection. On a non-extremal WIH, we want a preferred family of 2-sphere cross-sections that is defined purely from horizon-intrinsic data, is invariant under the horizon symmetry flow, and removes the remaining gauge freedom in choosing cross-sections.

Working with the rotational 1-form $\boldsymbol{\omega}$[1], we have the notion of a scalar potential whose exterior derivative,

$$\boldsymbol{\omega} \doteq -\star\, d\mathcal{U} + d\mathcal{V} \tag{3.3.3a}$$

where the Hodge dual is expanded in the indexed notation using the anti-symmetric area 1-form $\boldsymbol{\epsilon} = i\boldsymbol{m} \wedge \bar{\boldsymbol{m}}$,

$$\omega_\mu \doteq -\epsilon_\mu^{\ \nu} \hat{\nabla}_\nu \mathcal{U} + d\mathcal{V} \tag{3.3.3b}$$

where the potential $\mathcal{U}$ is determined by the rotational curvature scalar, $\operatorname{Im}\Psi_2$, that is related to the curl, $d\boldsymbol{\omega}$ on the horizon, and has an unambiguous projection onto the cross-section, and $\mathcal{V}$ relates to the divergence, represents the gauge freedom in the choice of $\boldsymbol{\omega}$, commonly setting $d\mathcal{V} = 0$. This unique foliation is always present on non-extremal WIHs and is defined in an invariant manner, meaning it can be constructed solely from structures that are inherently available on $(\mathscr{H}, [\boldsymbol{\ell}])$.

Assuming some natural constraints on the Ricci scalar, we arrive at the elliptic differential equation for the choice of $[\boldsymbol{\ell}]$, governed by the invertibility of the operator,

$$\boldsymbol{M} = \hat{\nabla}^2 + 2\boldsymbol{\omega}\cdot\hat{\nabla} + \hat{\nabla}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}^2 - \frac{1}{2}\,{}^{(2)}R + \frac{1}{2}\boldsymbol{h}\cdot\boldsymbol{R}. \tag{3.3.4}$$

The existence of a unique WIH foliation is determined if zero is not an eigenvalue of the elliptic operator, $\boldsymbol{M}$. Even if $\boldsymbol{M}$ has a non-trivial kernel, preferred $[\boldsymbol{\ell}]$s would exist but would not be unique.

[1] Hodge Decomposition Theorem says any smooth 1-form can be orthogonally decomposed into exact (longitudinal), co-exact (transversal), and harmonic. For a closed 1-form, we have the short Hodge decomposition into only the exact and harmonic terms. On the spherical cross-section, by the Hodge isomorphism theorem, we have no harmonic forms.

Further, in the absence of radiation, a suitable condition requires the expansion of $\boldsymbol{n}$ to be time-independent for generic NEHs that admit a unique $[\boldsymbol{\ell}]$ such that $(\mathscr{H}, [\boldsymbol{\ell}])$ is an IH. We don't have the natural freedom to choose the transversal expansion and shear as for WIHs, and the free data consists of $(\boldsymbol{h}, \boldsymbol{\omega}, \boldsymbol{S})$ on the cross section, and $\boldsymbol{h}$, $\boldsymbol{\omega}$ are subjected to further constraints.

3.4 Dynamical Horizons

Beyond the equilibrium case of IHs, we can define the dynamic nature of horizons that evolve over time, which are observed in nature. The general definition of a *Quasi-Local Horizon (QLH)* is a smooth submanifold $\mathscr{H}$ that is foliated by MOTS. It is a marginally trapped tube (MTT) that is formed by the union of MOTS, which evolve along the time parameter t.

As a particular case of QLHs, a *Dynamical Horizon (DH)* is a smooth spacelike surface $\mathscr{S}$, of codimension 1, which is foliated by compact submanifolds $\mathscr{T}_t$, which have codimension 2, and are marginally future trapped surfaces. A DH is thereby foliated by compact 2-surfaces such that, on each leaf of the foliation, the expansion of one null normal vanishes, and the expansion of the other null normal is strictly negative. The negative expansion of the null normal enables us to identify the inward-pointing null normal, and aligns with the second law of horizons, leading to an increase in the area of the trapped surfaces. Further, for DHs, the notion is *unique*, and independent of the choice of foliation $\mathscr{T}_t$, unlike NEHs.

Motivated by physical situations, involving strictly positive fluxes of energy across the horizon surface, we restrict to spacelike surfaces in the classical regime where the null energy conditions hold and energy density is positive. For the semi-classical regime, it is violated, as black holes can evaporate through emission of Hawking radiation. Here we define the DH as a timelike surface $\mathscr{S}$, which is foliated by compact 2-surfaces that are marginally trapped, and the expansion of the other null normal is positive everywhere.

Chapter 4

Probing Near Horizon Structure

For studying the geometry and specifying initial data, we apply the Characteristic Initial Value formulation, and further foliate the horizon $\mathscr{H}$ into submanifolds which are spacelike, characterised as the cross sections, $\mathscr{N}$. As previously, we use adapted null coordinates on the horizon described by the Newman-Penrose formalism. In the neighborhood of the horizon, we introduce coordinates similar to Gaussian null coordinates, or Bondi coordinates, in the neighborhood of null infinity. Let us first consider vacuum spacetimes and account for electromagnetic fields further.

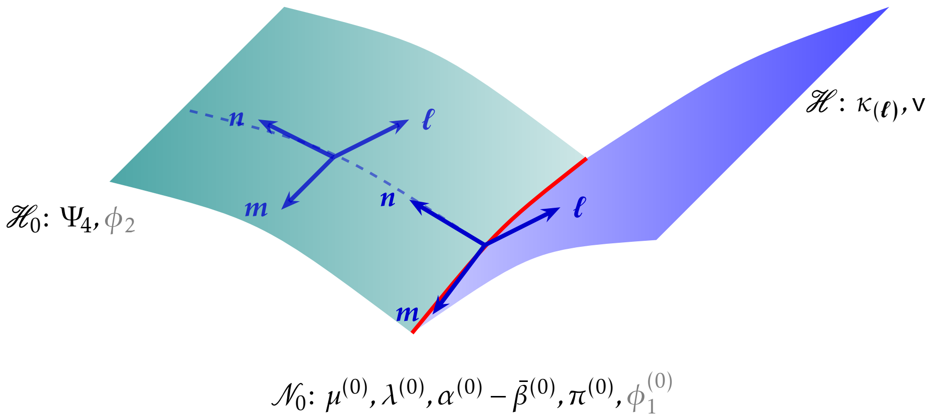

Figure 4.1: Representation of the characteristic initial value problem for an Isolated Horizon segment in a vacuum spacetime with adapted null coordinates following the Newman-Penrose formalism. The spacetime is the solution to a characteristic initial value problem with the initial data given on the horizon $\mathscr{H}$, and any null hypersurface, say $\mathscr{H}_0$, intersecting the horizon, with a spacelike cross section $\mathscr{N}_0$. The gray parameters indicate the inclusion of electromagnetic fields.

4.1 Near Horizon Geometry

Firstly, we use a function v, whose level sets give the leaves of the foliation of $\mathscr{H}$, and normalized by the relation connecting the appropriate 1-form $\underline{\boldsymbol{\ell}}$ and using the pull-back of the derivative operator (induced affine connection), $\hat{\nabla}$, relating $\boldsymbol{h}(\underline{\boldsymbol{\ell}}, \hat{\nabla}\mathrm{v}) \doteq 1$ for the induced metric $\boldsymbol{h}$ on the horizon. For the affine parameter v, let us denote the two surfaces (topological spheres) generated by the null generators as $\mathscr{N}_{\mathsf{v}}$ for constant v, where we introduce coordinates x^i for $i = 2, 3$, which we assert to be constant along $\boldsymbol{\ell}$, resulting in a coordinate system $(\mathsf{v}, \mathsf{x}^i)$ on $\mathscr{H}$.

The future-directed inward pointing null normal $\boldsymbol{n}$ is related to the foliation through the relation $\boldsymbol{n} \doteq -\nabla \mathsf{v}$, such that, we have $\boldsymbol{\ell} \cdot \boldsymbol{n} \doteq -1$. We extend $\boldsymbol{n}$ off geodesically, with r being an affine parameter along $-\boldsymbol{n}$, such that we set $\mathsf{r} = 0$ at the hypersurface $\mathscr{H}$. This yields a one-parameter family of null hypersurfaces $\mathscr{H}_{\mathsf{v}}$, parameterized by v, orthogonal to $\mathscr{N}_{\mathsf{v}}$. We set $(\mathsf{v}, \mathsf{x}^i)$ to be constant along the integral curves of $\boldsymbol{n}$ to obtain a coordinate system $(\mathsf{v}, \mathsf{r}, \mathsf{x}^i)$ in the neighbourhood of $\mathscr{H}$.

We complete the tetrad by adjoining a complex vector $\boldsymbol{m}$, tangent to some initial cross-section $\mathscr{N}_0$, such that we Lie drag $\boldsymbol{m}$ along $\boldsymbol{\ell}$,

$$\mathcal{L}_{\boldsymbol{\ell}} \boldsymbol{m} \doteq 0, \tag{4.1.1}$$

on $\mathscr{H}$. Due to the properties of a non-expanding horizon, Lie dragging preserves the normalization of the vectors. We parallel transport $\boldsymbol{\ell}$ and $\boldsymbol{m}$ along $-\boldsymbol{n}$, to obtain a null tetrad in the neighbourhood of $\mathscr{H}$. Thereby, we have the conditions on the directional derivatives,

$$\Delta \boldsymbol{n} = \Delta \boldsymbol{l} = \Delta \boldsymbol{m} = 0. \tag{4.1.2}$$

The construction is unique up to the choice of x^i and $\boldsymbol{m}$ on the initial cross section $\mathscr{N}_0$, and we are allowed to perform arbitrary spin transformations on $\mathscr{N}_0$. In the coordinates $(\mathsf{v}, \mathsf{r}, \mathsf{x}^i)$, we have the vectors on the null tetrad, given by,

$$\ell^i \nabla_i = D = \partial_{\mathsf{v}} + \mathsf{U} \partial_{\mathsf{r}} + \mathsf{X}^i \partial_{\mathsf{x}^i}, \tag{4.1.3a}$$

$$n^i \nabla_i = \Delta = -\partial_{\mathsf{r}}, \tag{4.1.3b}$$

$$m^i \nabla_i = \delta = \Omega \partial_{\mathsf{r}} + \xi^i \partial_{\mathsf{x}^i}, \tag{4.1.3c}$$

which are due to the constraints on the normalization conditions for the tetrad, given r is the affine parameter parallel propagating $\boldsymbol{n}$. Here the frame functions $\mathsf{U}, \mathsf{X}^i \in \mathbb{R}$ are real, and $\Omega, \xi \in \mathbb{C}$ are complex.

Since ∂_{v} is tangent to the horizon, with $\boldsymbol{\ell}$ as the null normal, at the horizon, we obtain,

$$\mathsf{U} \doteq 0, \tag{4.1.4a}$$

$$\mathsf{X}^i \doteq 0. \tag{4.1.4b}$$

Similarly, since we want $\boldsymbol{m}$ to be tangential to $\mathscr{N}_0$ at the horizon, we must ensure,

$$\Omega \doteq 0, \tag{4.1.4c}$$

by the nature of our construction.

4.1.1 Spin Coefficients

At all points in the neighbourhood of the horizon, we can constrain the spin coefficients using Eq. (4.1.2) since $\boldsymbol{n}$ is affinely paramterized, and $\boldsymbol{\ell}$, $\boldsymbol{m}$ are parallel propagated along $\boldsymbol{n}$, we use the relationships with the directional derivatives in Eq. (2.2.13), to arrive at,

$$\tau = \nu = \gamma = 0. \tag{4.1.5}$$

Using the above relations, we simplify the commutation relations in Eq. (2.2.23), which we write as,

$$[\Delta, D] = (\epsilon + \bar{\epsilon})\Delta - \pi\delta - \bar{\pi}\bar{\delta}, \tag{4.1.6a}$$
$$[\delta, D] = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta + (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}, \tag{4.1.6b}$$
$$[\delta, \Delta] = -(\bar{\alpha} + \beta)\Delta + \mu\delta + \bar{\lambda}\bar{\delta}, \tag{4.1.6c}$$
$$[\bar{\delta}, \delta] = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + a\delta + (\beta - \bar{\alpha})\bar{\delta}. \tag{4.1.6d}$$

Here, we apply it on the function v, with $D\mathsf{v} = 1$ from the normalization, and $\nabla\mathsf{v} = 0$, with $\delta\mathsf{v} = \bar{\delta}\mathsf{v} = 0$ due to coordinate independency of v and x^i, to have,

$$[\delta, D]\mathsf{v} = \bar{\alpha} + \beta - \bar{\pi} = 0, \tag{4.1.7a}$$
$$[\bar{\delta}, \delta]\mathsf{v} = \bar{\mu} - \mu = 0, \tag{4.1.7b}$$

Thus, we arrive at the additional relations for the spin coefficients which hold true in the neighbourhood of the horizon,

$$\mu = \bar{\mu}, \tag{4.1.8a}$$
$$\pi = \alpha + \bar{\beta}. \tag{4.1.8b}$$

In addition, we have the boundary condition at the horizon, specified by the nature of $\boldsymbol{\ell}$ being a geodesic, hypersurface-orthogonal and expansion-free, to arrive at

$$\kappa \doteq \rho \doteq 0, \tag{4.1.9}$$

due to Eq. (2.2.13), and the relations of the expansion and twist in Eq. (2.4.9) and Eq. (2.4.10d).

We can now employ the Raychaudhuri equation in Eq. (3.1.2a) applied to the null generators at the horizon, to realise the shear-free nature in Eq. (3.1.2b),

$$\sigma \doteq 0. \tag{4.1.10}$$

Further, the condition of Lie dragging $\boldsymbol{m}$ along $\boldsymbol{\ell}$ in Eq. (4.1.1), $\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{m} \doteq D\boldsymbol{m} - \delta\boldsymbol{\ell} \doteq (\epsilon - \bar{\epsilon})\boldsymbol{m} \doteq 0$ on the horizon by substituting Eq. (4.1.9), Eq. (4.1.10), results in a constraint on ϵ, as

$$\epsilon - \bar{\epsilon} \doteq 0, \tag{4.1.11}$$

which is a *gauge choice* which can always be made, and is employed further. This condition isn't a physical restriction but a convenient choice of frame, which is always possible to make via a spin transformation.

By the definition of an isolated horizon in Eq. (3.2.1), we have $\mathcal{L}_{\boldsymbol{\ell}}\boldsymbol{\omega} \doteq \boldsymbol{\ell}\cdot d\boldsymbol{\omega} + \hat{\nabla}\kappa_{(\boldsymbol{\ell})} \doteq 0$, for the rotation 1-form $\boldsymbol{\omega}$, where we have adapted to the tetrad on the horizon. We use the relation to the 1-form in Eq. (2.4.11), and substitute Eq. (4.1.8), to arrive at,

$$\boldsymbol{\omega} = (\epsilon+\bar{\epsilon})\boldsymbol{n} + \pi\boldsymbol{m} + \bar{\pi}\bar{\boldsymbol{m}}, \tag{4.1.12}$$

and the exterior derivative in Eq. (2.4.12) simplifies greatly using the constraints on the spin coefficients as,

$$d\boldsymbol{\omega} \doteq \operatorname{Im}\Psi_2\,\boldsymbol{\epsilon}, \tag{4.1.13}$$

where we denote the antisymmetric 1-form $\boldsymbol{\epsilon} = 2i\boldsymbol{m}\wedge\bar{\boldsymbol{m}}$. Thus, we require $\kappa_{(\boldsymbol{\ell})} \doteq \epsilon+\bar{\epsilon}$ as the surface gravity to be constant since $\hat{\nabla}\kappa_{(\boldsymbol{\ell})} \doteq 0$, due to the orthonormal relations of the tetrad, resulting in $\boldsymbol{\ell}\cdot d\boldsymbol{\omega} \doteq \operatorname{Im}\Psi_2\,\boldsymbol{\ell}\cdot\boldsymbol{\epsilon} \doteq 0$.

Further, we can use the commutation relations valid in the neighbourhood of the horizon, in Eq. (4.1.6), where we apply it on the function r, with the relation $\Delta\mathsf{r} = -1$ by the parameterization of $\boldsymbol{n}$ and using Eq. (4.1.3), to arrive at,

$$[\Delta, D]\mathsf{r} = \Delta\mathsf{U} = -\epsilon - \bar{\epsilon} - \pi\Omega - \bar{\pi}\bar{\Omega}, \tag{4.1.14a}$$

$$[\delta, D]\mathsf{r} = \delta\mathsf{U} - D\Omega = -\kappa + (\bar{\rho}+\epsilon-\bar{\epsilon})\Omega - \sigma\bar{\Omega}, \tag{4.1.14b}$$

$$[\delta, \Delta]\mathsf{r} = -\Delta\Omega = \pi + \mu\Omega + \bar{\lambda}\bar{\Omega}, \tag{4.1.14c}$$

$$[\bar{\delta}, \delta]\mathsf{r} = \bar{\delta}\Omega - \delta\bar{\Omega} = \rho - \bar{\rho} + a\Omega + (\beta - \bar{\alpha})\bar{\Omega}, \tag{4.1.14d}$$

and similarly for the function x^i, using the relation $\Delta\mathsf{x}^i = 0$, and imposing the relations above in Eq. (4.1.3), we have,

$$[\Delta, D]\mathsf{x}^i = \Delta\mathsf{X}^i = -\pi\xi^i - \bar{\pi}\bar{\xi}^i, \tag{4.1.14e}$$

$$[\delta, D]\mathsf{x}^i = \delta\mathsf{X}^i - D\xi^i = (\bar{\rho}+\epsilon-\bar{\epsilon})\xi^i - \sigma\bar{\xi}^i, \tag{4.1.14f}$$

$$[\delta, \Delta]\mathsf{x}^i = -\Delta\xi^i = \mu\xi^i + \bar{\lambda}\bar{\xi}^i, \tag{4.1.14g}$$

$$[\bar{\delta}, \delta]\mathsf{x}^i = \bar{\delta}\xi^i - \delta\bar{\xi}^i = a\xi^i + (\beta - \bar{\alpha})\bar{\xi}^i. \tag{4.1.14h}$$

We can combine the above equations, determining the radial and evolution equations for the spin coefficients of the tetrad, with the radial derivatives for the coefficients given by,

$$\Delta\mathsf{U} = -(\epsilon+\bar{\epsilon}) - \pi\Omega - \bar{\pi}\bar{\Omega}, \tag{4.1.15a}$$

$$\Delta\Omega = -\bar{\pi} - \mu\Omega - \bar{\lambda}\bar{\Omega}, \tag{4.1.15b}$$

$$\Delta\xi^i = -\mu\xi^i - \bar{\lambda}\bar{\xi}^i, \tag{4.1.15c}$$

$$\Delta \mathsf{X}^i = -\pi\xi^i - \bar{\pi}\bar{\xi}^i, \tag{4.1.15d}$$

while their propagation equations along v are,

$$D\Omega - \delta \mathsf{U} = \kappa - \rho\bar{\Omega} + \sigma\bar{\Omega}, \tag{4.1.15e}$$
$$D\xi^i - \delta \mathsf{X}^i = -\bar{\rho}\delta\xi^i + \sigma\bar{\xi}^i, \tag{4.1.15f}$$

where we have extended the gauge choice $\epsilon - \bar{\epsilon} = 0$, by the Lie dragging condition, to the neighbourhood of the horizon.

Now, we look into the field equations in Eq. (2.2.26), where we apply the requisite relations for the spin coefficients near the horizon in Eq. (4.1.5), and Eq. (4.1.8), and ignoring matter terms, to arrive at the the field equations involving radial derivatives,

$$\Delta\pi = -\mu\pi - \lambda\bar{\pi} - \Psi_3, \tag{4.1.16a}$$
$$\Delta\beta = -\beta\mu - \alpha\bar{\lambda}, \tag{4.1.16b}$$
$$\Delta\alpha = -\lambda\beta - \alpha\mu - \Psi_3, \tag{4.1.16c}$$
$$\Delta\lambda = -2\lambda\mu - \Psi_4, \tag{4.1.16d}$$
$$\Delta\mu = -\mu^2 - |\lambda|^2, \tag{4.1.16e}$$
$$\Delta\rho = -\rho\mu - \sigma\lambda - \Psi_2, \tag{4.1.16f}$$
$$\Delta\sigma = -\mu\sigma - \bar{\lambda}\rho, \tag{4.1.16g}$$
$$\Delta\kappa = -\rho\bar{\pi} - \sigma\pi - \Psi_1, \tag{4.1.16h}$$
$$\Delta\epsilon = -\alpha\bar{\pi} - \beta\pi - \Psi_2, \tag{4.1.16i}$$

where the radial equations decouple from the time evolution equations. Further, the time evolution equations containing the directional derivative D, become,

$$D\rho - \bar{\delta}\kappa = \rho^2 + |\sigma|^2 + \rho(\epsilon + \bar{\epsilon}) - 2\kappa\alpha, \tag{4.1.16j}$$
$$D\sigma - \delta\kappa = \sigma(\epsilon + \bar{\epsilon} + \rho + \bar{\rho}) - 2\kappa\beta + \Psi_0, \tag{4.1.16k}$$
$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda + \pi(\epsilon + \rho), \tag{4.1.16l}$$
$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa\mu + \epsilon(\bar{\pi} - \bar{\alpha}) + \Psi_1, \tag{4.1.16m}$$
$$D\lambda - \bar{\delta}\pi = \lambda(\rho - 2\epsilon) + \bar{\sigma}\mu + 2\pi\alpha, \tag{4.1.16n}$$
$$D\mu - \delta\pi = \mu(\bar{\rho} - \epsilon - \bar{\epsilon}) + \sigma\lambda + 2\pi\beta + \Psi_2, \tag{4.1.16o}$$

and the angular field equations comprising the 2-surface derivative δ, are,

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + |\alpha|^2 + |\beta|^2 - 2\alpha\beta - \Psi_2, \tag{4.1.16p}$$
$$\delta\rho - \bar{\delta}\sigma = \rho\bar{\pi} + \sigma\left(\bar{\beta} - 3\alpha\right) - \Psi_1, \tag{4.1.16q}$$
$$\delta\lambda - \bar{\delta}\mu = \mu\pi + \lambda(\bar{\alpha} - 3\beta) - \Psi_3. \tag{4.1.16r}$$

We have the Bianchi identities in Eq. (2.2.27), simplified extensively in the absence of matter in vacuum spacetime with the above constraints on the spin coefficients in Eq. (4.1.5), and Eq. (4.1.8), which can be split as the evolution equations given by,

$$D\Psi_1 - \bar{\delta}\Psi_0 = \Psi_0\left(\pi - 4\alpha\right) + 2\Psi_1\left(2\rho + \epsilon\right) - 3\kappa\Psi_2, \tag{4.1.17a}$$
$$D\Psi_2 - \bar{\delta}\Psi_1 = -\lambda\Psi_0 - 2\Psi_1\left(\pi - \alpha\right) + 3\rho\Psi_2 - 2\kappa\Psi_3, \tag{4.1.17b}$$
$$D\Psi_3 - \bar{\delta}\Psi_2 = -2\lambda\Psi_1 + 3\pi\Psi_2 - 2\Psi_3\left(\epsilon - \rho\right) - \kappa\Psi_4, \tag{4.1.17c}$$
$$D\Psi_4 - \bar{\delta}\Psi_3 = -3\lambda\Psi_2 + 2\Psi_3\left(2\pi + \alpha\right) - \Psi_4\left(4\epsilon - \rho\right), \tag{4.1.17d}$$

and the corresponding radial equations are extensively simplified as,

$$\Delta\Psi_0 - \delta\Psi_1 = -\mu\Psi_0 - 2\beta\Psi_1 + 3\sigma\Psi_2, \tag{4.1.17e}$$
$$\Delta\Psi_1 - \delta\Psi_2 = -2\mu\Psi_1 + 2\sigma\Psi_3, \tag{4.1.17f}$$
$$\Delta\Psi_2 - \delta\Psi_3 = -3\mu\Psi_2 + 2\beta\Psi_3 + \sigma\Psi_4, \tag{4.1.17g}$$
$$\Delta\Psi_3 - \delta\Psi_4 = -4\mu\Psi_3 + 4\beta\Psi_4. \tag{4.1.17h}$$

Since there exists no radial derivative component for Ψ_4, we can specify it as free data that we provide on the null hypersurface $\mathscr{H}_0$, as the past null cone originating from $\mathscr{N}_0$.

4.1.2 Radial Expansion

Having established the null tetrad, the rotation coefficients, the Weyl tensor, and the coordinate system, it is possible to solve the Einstein equations radially in the neighbourhood of $\mathscr{H}$. Precisely, we regard the spacetime as a solution to a characteristic initial value problem with the initial data given on the horizon $\mathscr{H}$, and any null hypersurface, say $\mathscr{H}_0$, intersecting the horizon.

Specifically, we expand the directional derivatives, rotation coefficients, and the components of the Weyl tensor, in a power series in r, away from the horizon, as

$$\mathsf{X} = \mathsf{X}^{(0)} + \mathsf{r}\mathsf{X}^{(1)} + \frac{1}{2}\mathsf{r}^2\mathsf{X}^{(2)} + \dots, \tag{4.1.18}$$

as a radial power series, such that we have the complex function $\mathsf{X} \doteq \mathsf{X}^{(0)}$ on the horizon.

In order to integrate the field equations and the Bianchi identities, we start with suitable data on some initial cross-section $\mathscr{N}_0$, of the horizon and use the non-radial equations to propagate them at all points of the horizon. Starting with this horizon data and the appropriate data on the past light cone $\mathscr{H}_0$ containing $\mathscr{N}_0$, the radial equations then yield the first radial derivatives and, iteratively, all successive higher derivatives as well.

Intrinsic Geometry

We first consider the intrinsic geometry of the horizon contained in the above equations at the horizon[1]. Using $\kappa^{(0)} = \rho^{(0)} = \sigma^{(0)} = 0$, we employ the radial field equation in Eq. (4.1.16), to arrive at,

$$\Psi_0^{(0)} = 0, \tag{4.1.19a}$$

on the hypersurface. Similarly, relating the vacuum angular field equation in Eq. (4.1.16), and substituting Eq. (4.1.5), Eq. (4.1.9) and Eq. (4.1.10), we arrive at,

$$\Psi_1^{(0)} = 0. \tag{4.1.19b}$$

Further, the field equation relating $D\pi^{(0)}$ in Eq. (4.1.16) on the horizon, is simplified as,

$$D\left(\alpha^{(0)} + \bar{\beta}^{(0)}\right) = \left(D\alpha^{(0)} - \bar{\delta}\epsilon^{(0)}\right) + \overline{\left(D\beta^{(0)} - \delta\epsilon^{(0)}\right)} + \bar{\delta}\left(\bar{\epsilon}^{(0)} + \epsilon^{(0)}\right) = 0, \tag{4.1.20a}$$

where we use the constraints for the spin coefficients carefully in Eq. (4.1.5), Eq. (4.1.9), Eq. (4.1.10) and Eq. (4.1.11) and $\epsilon^{(0)} + \bar{\epsilon}^{(0)}$ being constant.

Further, we can obtain the constancy of the connection coefficient $a^{(0)} = \alpha^{(0)} - \bar{\beta}^{(0)}$ compatible with the horizon metric $\boldsymbol{h}$, as

$$Da^{(0)} = \left(D\alpha^{(0)} - \bar{\delta}\epsilon^{(0)}\right) - \overline{\left(D\beta^{(0)} - \delta\epsilon^{(0)}\right)} - \bar{\delta}\left(\bar{\epsilon}^{(0)} - \epsilon^{(0)}\right) = 0. \tag{4.1.20b}$$

We can also use the relevant Bianchi equation in Eq. (4.1.17), to further arrive at a significant relationship suggesting the constancy of the geometry of the horizon, through,

$$D\Psi_2^{(0)} = 0. \tag{4.1.21}$$

The geometry can be realised further through the connection of the real component of $\Psi_2^{(0)}$ relating to the Gaussian curvature of the horizon, given by Eq. (2.4.5) in the vacuum spacetime, as

$${}^{(2)}R = -4\,\mathrm{Re}\,\Psi_2^{(0)}, \tag{4.1.22}$$

which is internally determined by the connection $a^{(0)}$ as in Eq. (2.4.4). Thereby, $\Psi_2^{(0)}$ is time-independent and its real component is determined by $a^{(0)}$, and the imaginary part corresponding to the curl of $\boldsymbol{\omega}$ in Eq. (4.1.13), which is determined by $\pi^{(0)} = \alpha^{(0)} + \bar{\beta}^{(0)}$.

[1] We compute the $\mathcal{O}(1)$ component of the equations, and shall often drop the index on the directional derivatives to avoid clutter, when the relevant order of the operator is the same as the order of the operand.

We can also derive $\Psi_3^{(0)}$ from the angular field equations in Eq. (4.1.16), such that we express in terms of the spin operator $\eth$ for the angular derivatives, to arrive at,

$$\Psi_3^{(0)} = \bar{\eth}\mu^{(0)} - \eth\lambda^{(0)} + \left(\mu^{(0)} - \lambda^{(0)}\right)\pi^{(0)}, \tag{4.1.23}$$

where we have utilized the spin-weights of μ as $\mathfrak{s} = 0$, and for λ to be $\mathfrak{s} = 1$ as in Eq. (2.3.10)

The other interesting field equations in Eq. (4.1.16), lead us to the time evolution of the expansion and shear of $\boldsymbol{n}$ at the horizon,

$$D\lambda^{(0)} + \kappa_{(\boldsymbol{\ell})}\lambda^{(0)} = \bar{\eth}\pi^{(0)} + \left(\pi^{(0)}\right)^2, \tag{4.1.24a}$$

$$D\mu^{(0)} + \kappa_{(\boldsymbol{\ell})}\mu^{(0)} = \eth\pi^{(0)} + \left|\pi^{(0)}\right|^2 + \Psi_2^{(0)}, \tag{4.1.24b}$$

which is rewritten in terms of the spin operator, with the spin-weight of π as $\mathfrak{s} = -1$ as in Eq. (2.3.10).

We have the time evolution of the transversal expansion $\mu^{(0)}$ and shear $\lambda^{(0)}$ of $\boldsymbol{n}$, given by the solution to the above differential equations, as

$$\mu^{(0)}(\mathsf{v}) = \mu^{(0)}(0)\exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right) + \frac{1}{\kappa_{(\boldsymbol{\ell})}}\left(\bar{\eth}\pi^{(0)} + \left(\pi^{(0)}\right)^2\right)\left(1 - \exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right)\right), \tag{4.1.25a}$$

$$\lambda^{(0)}(\mathsf{v}) = \lambda^{(0)}(0)\exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right) + \frac{1}{\kappa_{(\boldsymbol{\ell})}}\left(\eth\pi^{(0)} + \left|\pi^{(0)}\right|^2 + \Psi_2^{(0)}\right)\left(1 - \exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right)\right), \tag{4.1.25b}$$

where $\mu^{(0)}(0)$ and $\lambda^{(0)}(0)$ are initial data to be specified on $\mathscr{N}_0$.

Through the relationship with $\Psi_3^{(0)}$ in Eq. (4.1.23), we can determine its time evolution, which simplifies as,

$$\Psi_3^{(0)}(\mathsf{v}) = \Psi_3^{(0)}(0)\exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right) + \frac{1}{\kappa_{(\boldsymbol{\ell})}}\left(\eth\Psi_2^{(0)} + 3\pi^{(0)}\Psi_2^{(0)}\right)\left(1 - \exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right)\right), \tag{4.1.26}$$

where $\Psi_3^{(0)}(0)$ is completely determined by $\mu^{(0)}(0)$ and $\lambda^{(0)}$.

Further, the other Bianchi identities in Eq. (4.1.17) lead us to the evolution equation of Ψ_4 at the horizon as,

$$D\Psi_4^{(0)} + 2\kappa_{(\boldsymbol{\ell})}\Psi_4^{(0)} = \bar{\eth}\Psi_3^{(0)} - 3\lambda^{(0)}\Psi_2^{(0)} + 5\pi^{(0)}\Psi_3^{(0)}, \tag{4.1.27}$$

where we have utilized the spin-weight of Ψ_3 as $\mathfrak{s} = -1$ given by Eq. (2.3.11). We realise the nature of the first-order linear ordinary differential equation for $\Psi_4^{(0)}(\mathsf{v})$ and we set,

$$\left(\bar{\eth} + 5\pi^{(0)}\right)\Psi_3^{(0)} - 3\lambda^{(0)}\Psi_2^{(0)} = \mathsf{A} + \mathsf{B}\exp\left(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\right), \tag{4.1.28a}$$

such that we can solve the equation,

$$\Psi_4^{(0)}(\mathsf{v}) = \Psi_4^{(0)}(0) + \frac{\mathsf{A}}{2\kappa_{(\boldsymbol{\ell})}}\Big(1 - \exp\Big(-2\kappa_{(\boldsymbol{\ell})}\mathsf{v}\Big)\Big) + \frac{\mathsf{B}}{\kappa_{(\boldsymbol{\ell})}}\Big(1 - \exp\Big(-\kappa_{(\boldsymbol{\ell})}\mathsf{v}\Big)\Big). \tag{4.1.28b}$$

where the angular time-independent functions determined by

$$\mathsf{A} = \frac{1}{\kappa_{(\boldsymbol{\ell})}}\Big(\Big(\bar{\eth} + 5\pi^{(0)}\Big)\Big(\bar{\eth} + 3\pi^{(0)}\Big) - 3\Big(\eth\pi^{(0)} + \Big|\pi^{(0)}\Big|^2 + \Psi_2^{(0)}\Big)\Big)\Psi_2^{(0)}, \tag{4.1.28c}$$

$$\mathsf{B} = \Big(\bar{\eth} + 5\pi^{(0)}\Big)\Psi_3^{(0)}(0) - 3\lambda^{(0)}\Psi_2^{(0)} - \mathsf{A}. \tag{4.1.28d}$$

Thereby, due to the geometry remaining constant on the initial cross-section $\mathscr{N}_0$, we start with a choice of the null generator $\boldsymbol{l}$ and specify a constant $\kappa_{(\boldsymbol{\ell})} = \epsilon^{(0)} + \bar{\epsilon}^{(0)}$ for the surface gravity, and an affine parameter v. For surfaces with constant v, given by $\mathscr{N}_{\mathsf{v}}$, we choose a basis $\{\boldsymbol{m}, \bar{\boldsymbol{m}}\}$, which are tangent to $\mathscr{N}_{\mathsf{v}}$ and are Lie dragged along $\boldsymbol{\ell}$. On one of the 2-surfaces, say $\mathscr{N}_0$, we choose the transversal expansion $\mu^{(0)}$, shear $\lambda^{(0)}$, the spin coefficient $\pi^{(0)}$, the 2-surface connection coefficient $a^{(0)}$ (which is time independent, given by Eq. (4.1.20b)), and the transverse gravitational radiation $\Psi_4^{(0)}$.

Radial Evolution

The radial derivatives can be calculated sequentially order-by-order. By substituting the relevant spin coefficients of the intrinsic horizon geometry in the field equations involving radial derivatives in Eq. (4.1.16), we can obtain the first radial derivatives describing their radial evolution, as,

$$\pi^{(1)} = -\Delta\pi = \mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} + \Psi_3^{(0)}, \tag{4.1.29a}$$

$$\beta^{(1)} = -\Delta\beta = \beta^{(0)}\mu^{(0)} + \alpha^{(0)}\bar{\lambda}^{(0)}, \tag{4.1.29b}$$

$$\alpha^{(1)} = -\Delta\alpha = \lambda^{(0)}\beta^{(0)} + \alpha^{(0)}\mu^{(0)} + \Psi_3^{(0)}, \tag{4.1.29c}$$

$$\lambda^{(1)} = -\Delta\lambda = 2\lambda^{(0)}\mu^{(0)} + \Psi_4^{(0)}, \tag{4.1.29d}$$

$$\mu^{(1)} = -\Delta\mu = \Big(\mu^{(0)}\Big)^2 + \Big|\lambda^{(0)}\Big|^2, \tag{4.1.29e}$$

$$\rho^{(1)} = -\Delta\rho = \Psi_2^{(0)}, \tag{4.1.29f}$$

$$\sigma^{(1)} = -\Delta\sigma = 0, \tag{4.1.29g}$$

$$\kappa^{(1)} = -\Delta\kappa = 0, \tag{4.1.29h}$$

$$\epsilon^{(1)} = -\Delta\epsilon = \alpha^{(0)}\bar{\pi}^{(0)} + \beta^{(0)}\pi^{(0)} + \Psi_2^{(0)}. \tag{4.1.29i}$$

Thereby, the interesting quantities relating the radial evolution of the connection coefficient a, and the surface gravity $\kappa_{(\boldsymbol{\ell})}$, are,

$$\epsilon^{(1)} + \bar{\epsilon}^{(1)} = \kappa_{(\boldsymbol{\ell})}^{(1)} = 2\Big|\pi^{(0)}\Big|^2 + 2\,\mathrm{Re}\,\Psi_2^{(0)}, \tag{4.1.30a}$$

$$\epsilon^{(1)} - \bar{\epsilon}^{(1)} = \bar{\pi}^{(0)} a^{(0)} - \pi^{(0)} \bar{a}^{(0)} + 2i\,\mathrm{Im}\,\Psi_2^{(0)}, \tag{4.1.30b}$$

$$a^{(1)} = \mu^{(0)} a^{(0)} - \lambda^{(0)} \bar{a}^{(0)} + \Psi_3^{(0)}. \tag{4.1.30c}$$

Further, the radial Bianchi identities in Eq. (4.1.17) determined the radial evolution of the components of the projections of the Weyl tensors as,

$$\Psi_0^{(1)} = -\Delta\Psi_0 = 0, \tag{4.1.31a}$$

$$\Psi_1^{(1)} = -\Delta\Psi_1 = -\delta\Psi_2^{(0)} = -\eth\Psi_2^{(0)}, \tag{4.1.31b}$$

$$\begin{aligned}\Psi_2^{(1)} &= -\Delta\Psi_2 = -\delta\Psi_3^{(0)} + 3\mu^{(0)}\Psi_2^{(0)} - 2\beta^{(0)}\Psi_3^{(0)} \\ &= -\left(\eth + \bar{\pi}^{(0)}\right)\Psi_3^{(0)} + 3\mu^{(0)}\Psi_2^{(0)},\end{aligned} \tag{4.1.31c}$$

$$\begin{aligned}\Psi_3^{(1)} &= -\Delta\Psi_3 = -\delta\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)} + 4\beta^{(0)}\Psi_4^{(0)} \\ &= -\left(\eth + 2\bar{\pi}^{(0)}\right)\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)}\end{aligned} \tag{4.1.31d}$$

Iteratively, we can now evaluate the expressions for the second-order derivatives by applying the radial derivative in Eq. (4.1.16), and substituting the first derivatives, to arrive at,

$$\begin{aligned}\pi^{(2)} = \Delta^2\pi &= -\mu\Delta\pi - \pi\Delta\mu - \lambda\Delta\bar{\pi} - \bar{\pi}\Delta\lambda - \Delta\Psi_3 \\ &= \pi^{(0)}\mu^{(1)} + \mu^{(0)}\pi^{(1)} + \lambda^{(0)}\bar{\pi}^{(1)} + \Psi_3^{(1)} \\ &= 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2\right)\pi^{(0)} + 4\bar{\pi}^{(0)}\lambda^{(0)}\mu^{(0)} + 5\mu^{(0)}\Psi_3^{(0)} \\ &\quad + \lambda^{(0)}\bar{\Psi}_3^{(0)} - \eth\Psi_4^{(0)} - \bar{\pi}^{(0)}\Psi_4^{(0)},\end{aligned} \tag{4.1.32a}$$

$$\begin{aligned}\beta^{(2)} = \Delta^2\beta &= -\beta\Delta\mu - \mu\Delta\beta - \alpha\Delta\bar{\lambda} - \bar{\lambda}\Delta\alpha \\ &= \beta^{(0)}\mu^{(1)} + \mu^{(0)}\beta^{(1)} + \alpha^{(0)}\bar{\lambda}^{(1)} + \bar{\lambda}^{(0)}\alpha^{(1)} \\ &= 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2\right)\beta^{(0)} + 4\mu^{(0)}\bar{\lambda}^{(0)}\alpha^{(0)} + \alpha^{(0)}\bar{\Psi}_4^{(0)} + \bar{\lambda}^{(0)}\Psi_3^{(0)},\end{aligned} \tag{4.1.32b}$$

$$\begin{aligned}\alpha^{(2)} = \Delta^2\alpha &= -\lambda\Delta\beta - \beta\Delta\lambda - \alpha\Delta\mu - \mu\Delta\alpha - \Delta\Psi_3 \\ &= \lambda^{(0)}\beta^{(1)} + \beta^{(0)}\lambda^{(1)} + \alpha^{(0)}\mu^{(1)} + \mu^{(0)}\alpha^{(1)} + \Psi_3^{(1)} \\ &= 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2\right)\alpha^{(0)} + 4\mu^{(0)}\lambda^{(0)}\beta^{(0)} + \mu^{(0)}\Psi_3^{(0)} - \eth\Psi_4^{(0)} \\ &\quad - 2\bar{\pi}^{(0)}\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)},\end{aligned} \tag{4.1.32c}$$

$$\begin{aligned}\lambda^{(2)} = \Delta^2\lambda &= -2\lambda\Delta\mu - 2\mu\Delta\lambda - \Delta\Psi_4 \\ &= 2\lambda^{(0)}\mu^{(1)} + 2\mu^{(0)}\lambda^{(1)} + \Psi_4^{(1)} \\ &= 6\lambda^{(0)}\left(\mu^{(0)}\right)^2 + 2\lambda^{(0)}\left|\lambda^{(0)}\right|^2 + 2\mu^{(0)}\Psi_4^{(0)} + \Psi_4^{(1)},\end{aligned} \tag{4.1.32d}$$

$$\begin{aligned}
\mu^{(2)} = \Delta^2\mu &= -2\mu\Delta\mu - \bar{\lambda}\Delta\lambda - \lambda\Delta\bar{\lambda} \\
&= 2\mu^{(0)}\mu^{(1)} + \bar{\lambda}^{(0)}\lambda^{(1)} + \lambda^{(0)}\bar{\lambda}^{(1)} \\
&= 2\left(\mu^{(0)}\right)^3 + 6\mu^{(0)}\left|\lambda^{(0)}\right|^2 + \lambda^{(0)}\bar{\Psi}_4^{(0)} + \bar{\lambda}^{(0)}\Psi_4^{(0)}, \qquad (4.1.32e) \\
\rho^{(2)} = \Delta^2\rho &= -\rho\Delta\mu - \mu\Delta\rho - \sigma\Delta\lambda - \lambda\Delta\sigma - \Delta\Psi_2 \\
&= \rho^{(0)}\mu^{(1)} + \mu^{(0)}\rho^{(1)} + \sigma^{(0)}\lambda^{(1)} + \lambda^{(0)}\sigma^{(1)} + \Psi_2^{(1)} \\
&= 4\mu^{(0)}\Psi_2^{(0)} - \eth\Psi_3^{(0)} - \bar{\pi}^{(0)}\Psi_3^{(0)}, \qquad (4.1.32f) \\
\sigma^{(2)} = \Delta^2\sigma &= -\mu\Delta\sigma - \sigma\Delta\mu - \bar{\lambda}\Delta\rho - \rho\Delta\bar{\lambda} \\
&= \mu^{(0)}\sigma^{(1)} + \sigma^{(0)}\mu^{(1)} + \bar{\lambda}^{(0)}\rho^{(1)} + \rho^{(0)}\bar{\lambda}^{(1)} \\
&= \bar{\lambda}^{(0)}\Psi_2^{(0)}, \qquad (4.1.32g) \\
\kappa^{(2)} = \Delta^2\kappa &= -\rho\Delta\bar{\pi} - \bar{\pi}\Delta\rho - \sigma\Delta\pi - \pi\Delta\sigma - \Delta\Psi_1 \\
&= \rho^{(0)}\bar{\pi}^{(1)} + \bar{\pi}^{(0)}\rho^{(1)} + \sigma^{(0)}\pi^{(1)} + \pi^{(0)}\sigma^{(1)} + \Psi_1^{(1)} \\
&= -\eth\Psi_2^{(0)} - \bar{\pi}^{(0)}\Psi_2^{(0)}, \qquad (4.1.32h) \\
\epsilon^{(2)} = \Delta^2\epsilon &= -\alpha\Delta\bar{\pi} - \bar{\pi}\Delta\alpha - \beta\Delta\pi - \pi\Delta\beta - \Delta\Psi_2 \\
&= \alpha^{(0)}\bar{\pi}^{(1)} + \bar{\pi}^{(0)}\alpha^{(1)} + \beta^{(0)}\pi^{(1)} + \pi^{(0)}\beta^{(1)} + \Psi_2^{(1)} \\
&= 2\left(\alpha^{(0)} + \beta^{(0)}\right)\mu^{(0)}\bar{\pi}^{(0)} + 2\pi^{(0)}\alpha^{(0)}\bar{\lambda}^{(0)} + 2\bar{\pi}^{(0)}\beta^{(0)}\lambda^{(0)} + \alpha^{(0)}\bar{\Psi}_3^{(0)} \\
&\quad + \beta^{(0)}\Psi_3^{(0)} - \eth\Psi_3^{(0)} + 3\mu^{(0)}\Psi_2^{(0)}. \qquad (4.1.32i)
\end{aligned}$$

Further, the second-order radial evolution of the connection coefficient a, and the surface gravity $\kappa_{(\boldsymbol{\ell})}$, are given by,

$$\begin{aligned}
\epsilon^{(2)} + \bar{\epsilon}^{(2)} = \kappa_{(\boldsymbol{\ell})}^{(2)} &= 4\mu^{(0)}\left|\pi^{(0)}\right|^2 + 2\lambda^{(0)}\left(\bar{\pi}^{(0)}\right)^2 + 2\bar{\lambda}^{(0)}\left(\pi^{(0)}\right)^2 + \pi^{(0)}\bar{\Psi}_3^{(0)} \\
&\quad + \bar{\pi}^{(0)}\Psi_3^{(0)} - \eth\Psi_3^{(0)} - \bar{\eth}\bar{\Psi}_3^{(0)} + 6\mu^{(0)}\,\mathrm{Re}\,\Psi_2^{(0)}, \qquad (4.1.33a) \\
\epsilon^{(2)} - \bar{\epsilon}^{(2)} &= 2\mu^{(0)}\left(\left(\alpha^{(0)} + \beta^{(0)}\right)\bar{\pi}^{(0)} - \left(\bar{\alpha}^{(0)} + \bar{\beta}^{(0)}\right)\pi^{(0)}\right) + 2ia^{(0)}\,\mathrm{Im}\,\Psi_3^{(0)} \\
&\quad + 2a^{(0)}\left(2\pi^{(0)}\bar{\lambda}^{(0)} - 2\bar{\pi}^{(0)}\lambda^{(0)}\right) - \eth\Psi_3^{(0)} + \bar{\eth}\bar{\Psi}_3^{(0)} \\
&\quad + 6i\mu^{(0)}\,\mathrm{Im}\,\Psi_2^{(0)}, \qquad (4.1.33b) \\
a^{(2)} &= -2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 - 2\mu^{(0)}\lambda^{(0)} + \frac{1}{2}\Psi_4^{(0)}\right)\bar{a}^{(0)} + 5\mu^{(0)}\Psi_3^{(0)} \\
&\quad - \lambda^{(0)}\bar{\Psi}_3^{(0)} - \eth\Psi_4^{(0)} - 2\bar{\pi}^{(0)}\Psi_4^{(0)}. \qquad (4.1.33c)
\end{aligned}$$

Expansion of the Metric

We now employ the radial equations in Eq. (4.1.15) for evaluating the frame functions, with $\mathsf{U}^{(0)} = \mathsf{X}^{(0),i} = \Omega^{(0)} = 0$ from Eq. (4.1.4), to have,

$$\mathsf{U}^{(1)} = -\Delta \mathsf{U} = \epsilon^{(0)} + \bar{\epsilon}^{(0)} = \kappa_{(\boldsymbol{\ell})}, \tag{4.1.34a}$$

$$\Omega^{(1)} = -\Delta\Omega = \bar{\pi}^{(0)}, \tag{4.1.34b}$$

$$\xi^{(1),i} = -\Delta\xi^{i} = \mu^{(0)}\xi^{(0),i} + \bar{\lambda}^{(0)}\bar{\xi}^{(0),i}, \tag{4.1.34c}$$

$$\mathsf{X}^{(1),i} = -\Delta \mathsf{X}^{i} = \pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}, \tag{4.1.34d}$$

where we have substituted the spin coefficients on the horizon.

Thus, we compute the directional derivatives, expanding up to first-order in r, using the first-order frame functions in Eq. (4.1.3), to have

$$D = \partial_{\mathsf{v}} + \mathsf{r}\kappa_{(\boldsymbol{\ell})}\partial_{\mathsf{r}} + \mathsf{r}\left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right)\partial_{\mathsf{x}^{i}} + \dots, \tag{4.1.35a}$$

$$\Delta = -\partial_{\mathsf{r}}, \tag{4.1.35b}$$

$$\delta = \xi^{(0),i}\partial_{\mathsf{x}^{i}} + \mathsf{r}\bar{\pi}^{(0)}\partial_{\mathsf{r}} + \mathsf{r}\left(\mu^{(0)}\xi^{(0),i} + \bar{\lambda}^{(0)}\bar{\xi}^{(0),i}\right)\partial_{\mathsf{x}^{i}} + \dots, \tag{4.1.35c}$$

Similarly, in the second-order, we apply the radial derivative operator to Eq. (4.1.15), to arrive at,

$$\begin{aligned}
\mathsf{U}^{(2)} = \Delta^{2}\mathsf{U} &= -(\Delta\epsilon + \Delta\bar{\epsilon}) - \pi\Delta\Omega - \Omega\Delta\pi - \bar{\pi}\Delta\bar{\Omega} - \bar{\Omega}\Delta\bar{\pi} \\
&= \epsilon^{(1)} + \bar{\epsilon}^{(1)} + \pi^{(0)}\Omega^{(1)} + \Omega^{(0)}\pi^{(1)} + \bar{\pi}^{(0)}\bar{\Omega}^{(1)} + \bar{\Omega}^{(0)}\bar{\pi}^{(1)} \\
&= 4\left|\pi^{(0)}\right|^{2} + 2\,\mathrm{Re}\,\Psi_{2}^{(0)},
\end{aligned} \tag{4.1.36a}$$

$$\begin{aligned}
\Omega^{(2)} = \Delta^{2}\Omega &= -\Delta\pi - \mu\Delta\Omega - \Omega\Delta\mu - \bar{\lambda}\Delta\bar{\Omega} - \bar{\Omega}\Delta\bar{\lambda} \\
&= \pi^{(1)} + \mu^{(0)}\Omega^{(1)} + \Omega^{(0)}\mu^{(1)} + \bar{\lambda}^{(0)}\bar{\Omega}^{(1)} + \bar{\Omega}^{(0)}\bar{\lambda}^{(1)} \\
&= 2\mu^{(0)}\bar{\pi}^{(0)} + 2\bar{\lambda}^{(0)}\pi^{(0)} + \bar{\Psi}_{3}^{(0)},
\end{aligned} \tag{4.1.36b}$$

$$\begin{aligned}
\xi^{(2),i} = \Delta^{2}\xi^{i} &= -\mu\Delta\xi^{i} - \xi^{i}\Delta\mu - \bar{\lambda}\Delta\bar{\xi}^{i} - \bar{\xi}^{i}\Delta\bar{\lambda} \\
&= \mu^{(0)}\xi^{(1),i} + \xi^{(0),i}\mu^{(1)} + \bar{\lambda}^{(0)}\bar{\xi}^{(1),i} + \bar{\xi}^{(0),i}\bar{\lambda}^{(1)} \\
&= 2\left(\left(\mu^{(0)}\right)^{2} + \left|\lambda^{(0)}\right|^{2}\right)\xi^{(0),i} + \left(4\mu^{(0)}\bar{\lambda}^{(0)} + \bar{\Psi}_{4}^{(0)}\right)\bar{\xi}^{(0),i},
\end{aligned} \tag{4.1.36c}$$

$$\begin{aligned}
\mathsf{X}^{(2),i} = \Delta^{2}\mathsf{X}^{i} &= -\pi\Delta\xi^{i} - \xi^{i}\Delta\pi - \bar{\pi}\Delta\bar{\xi}^{i} - \bar{\xi}^{i}\Delta\bar{\pi} \\
&= \pi^{(0)}\xi^{(1),i} + \xi^{(0),i}\pi^{(1)} + \bar{\pi}^{(0)}\bar{\xi}^{(1),i} + \bar{\xi}^{(0),i}\bar{\pi}^{(1)} \\
&= 2\mu^{(0)}\left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right) + 2\lambda^{(0)}\bar{\pi}^{(0)}\xi^{(0),i} \\
&\quad + 2\bar{\lambda}^{(0)}\pi^{(0)}\bar{\xi}^{(0),i} + \Psi_{3}^{(0)}\xi^{(0),i} + \bar{\Psi}_{3}^{(0)}\bar{\xi}^{(0),i}.
\end{aligned} \tag{4.1.36d}$$

Thereby, the frame functions can be expanded in the radial direction, substituting the first and second-order derivatives from Eq. (4.1.34) and Eq. (4.1.36), as

$$\mathsf{U} = \kappa_{(\boldsymbol{\ell})}\mathsf{r} + \left(2\left|\pi^{(0)}\right|^2 + \mathrm{Re}\,\Psi_2^{(0)}\right)\mathsf{r}^2 + \dots, \tag{4.1.37a}$$

$$\Omega = \bar{\pi}^{(0)}\mathsf{r} + \left(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} + \frac{1}{2}\bar{\Psi}_3^{(0)}\right)\mathsf{r}^2 + \dots, \tag{4.1.37b}$$

$$\xi^i = \xi^{(0),i} + \left(\mu^{(0)}\xi^{(0),i} + \bar{\lambda}^{(0)}\bar{\xi}^{(0),i}\right)\mathsf{r} + \left(\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2\right)\xi^{(0),i} + \left(2\mu^{(0)}\bar{\lambda}^{(0)} + \frac{1}{2}\bar{\Psi}_4^{(0)}\right)\bar{\xi}^{(0),i}\right)\mathsf{r}^2 + \dots, \tag{4.1.37c}$$

$$\begin{aligned}\mathsf{X}^i &= \left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right)\mathsf{r} + \left(\left(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} + \frac{1}{2}\Psi_3^{(0)}\right)\xi^{(0),i}\right.\\ &\quad \left.+ \left(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} + \frac{1}{2}\bar{\Psi}_3^{(0)}\bar{\xi}^{(0),i}\right)\right)\mathsf{r}^2 + \dots \qquad (4.1.37\mathrm{d})\\ &= \Omega\bar{\xi}^{(0),i} + \bar{\Omega}\xi^{(0),i} + \dots, \qquad (4.1.37\mathrm{e})\end{aligned}$$

which can be used analogously to compute the directional derivatives.

The contravariant spacetime metric $\boldsymbol{g}$, as expressed in the Newman-Penrose formalism, $g^{\mu\nu} = 2\left(-\ell^{(\mu}n^{\nu)} + m^{(\mu}\bar{m}^{\nu)}\right)$ in Eq. (2.2.8), can be related to the frame functions, to have,

$$\begin{aligned}g^{\mathsf{rr}} &= 2\left(\mathsf{U} + |\Omega|^2\right) \qquad (4.1.38\mathrm{a})\\ &= 2\kappa_{(\boldsymbol{\ell})}\mathsf{r} + 2\left(3\left|\pi^{(0)}\right|^2 + \mathrm{Re}\,\Psi_2^{(0)}\right)\mathsf{r}^2 + \dots, \qquad (4.1.38\mathrm{b})\end{aligned}$$

$$g^{\mathsf{vr}} = 1, \tag{4.1.38c}$$

$$\begin{aligned}g^{\mathsf{r}\mathsf{x}^i} &= \mathsf{X}^i + \bar{\Omega}\xi^i + \Omega\bar{\xi}^i \qquad (4.1.38\mathrm{d})\\ &= 2\left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right)\mathsf{r} + 2\left(\left(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} + \frac{1}{2}\Psi_3^{(0)}\right)\xi^{(0),i}\right.\\ &\quad \left.+ \left(\mu^{(0)}\bar{\pi}^{(0)} + \lambda^{(0)}\pi^{(0)} + \frac{1}{2}\bar{\Psi}_3^{(0)}\right)\bar{\xi}^{(0),i}\right)\mathsf{r}^2 + \dots, \qquad (4.1.38\mathrm{e})\end{aligned}$$

$$\begin{aligned}g^{\mathsf{x}^i\mathsf{x}^j} &= \xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j \qquad (4.1.38\mathrm{f})\\ &= \left(\xi^{(0),i}\bar{\xi}^{(0),j} + \xi^{(0),j}\bar{\xi}^{(0),i}\right) + 2\left(\mu^{(0)}\left(\xi^{(0),i}\bar{\xi}^{(0),j} + \xi^{(0),j}\bar{\xi}^{(0),i}\right)\right.\\ &\quad \left.+ \lambda^{(0)}\xi^{(0),i}\xi^{(0),j} + \bar{\lambda}^{(0)}\bar{\xi}^{(0),i}\bar{\xi}^{(0),j}\right)\mathsf{r} + \left(3\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2\right)\right.\\ &\quad \times\left(\xi^{(0),i}\bar{\xi}^{(0),j} + \xi^{(0),j}\bar{\xi}^{(0),i}\right) + \left(6\mu^{(0)}\lambda^{(0)} + \Psi_4^{(0)}\right)\xi^{(0),i}\xi^{(0),j}\\ &\quad \left.+ \left(6\mu^{(0)}\bar{\lambda}^{(0)} + \bar{\Psi}_4^{(0)}\right)\bar{\xi}^{(0),i}\bar{\xi}^{(0),j}\right)\mathsf{r}^2 + \dots, \qquad (4.1.38\mathrm{g})\end{aligned}$$

where we have used the relations with the directional derivatives and the frame fields in Eq. (4.1.4), and have expanded the frame functions using Eq. (4.1.37).

The basis 1-forms dual to the null tetrad in Eq. (4.1.3), can thereby be obtained,

$$\begin{aligned}\boldsymbol{\ell} &= d\mathsf{r} - \left(\kappa_{(\boldsymbol{\ell})}\mathsf{r} + \mathrm{Re}\,\Psi_2^{(0)}\mathsf{r}^2\right)d\mathsf{v} - \left(\pi^{(0)}\mathsf{r} + \frac{1}{2}\Psi_3^{(0)}\mathsf{r}^2\right)\xi_i^{(0)}d\mathsf{x}^i \\ &\quad - \left(\bar{\pi}^{(0)}\mathsf{r} + \frac{1}{2}\bar{\Psi}_3^{(0)}\mathsf{r}^2\right)\bar{\xi}_i^{(0)}d\mathsf{x}^i + \dots, &(4.1.39\text{a}) \\ \boldsymbol{n} &= -d\mathsf{v}, &(4.1.39\text{b}) \\ \boldsymbol{m} &= -\left(\bar{\pi}^{(0)}\mathsf{r} + \frac{1}{2}\Psi_3^{(0)}\mathsf{r}^2\right)d\mathsf{v} + \left(1 - \mu^{(0)}\mathsf{r}\right)\xi_i^{(0)}d\mathsf{x}^i - \left(\bar{\lambda}^{(0)}\mathsf{r} + \frac{1}{2}\bar{\Psi}_4^{(0)}\mathsf{r}^2\right)\bar{\xi}_i^{(0)}d\mathsf{x}^i + \dots, \\ & &(4.1.39\text{c})\end{aligned}$$

where we have defined $\xi_i^{(0)}$ as the dual, through the relations $\xi_i^{(0)}\xi^{(0),i} = 0$ and $\xi_i^{(0)}\bar{\xi}^{(0),i} = 1$, such that, we can define $m_\mu^{(0)} = \xi_i^{(0)}\partial_\mu \mathsf{x}^i$.

For the covariant metric, we can expand the relation in Eq. (2.2.9), to arrive at,

$$g_{\mu\nu} = 2\left(-\ell_{(\mu}n_{\nu)} + m_{(\mu}\bar{m}_{\nu)}\right) = g_{\mu\nu}^{(0)} + \mathsf{r}g_{\mu\nu}^{(1)} + \frac{1}{2}\mathsf{r}^2 g_{\mu\nu}^{(0)} + \dots, \tag{4.1.40}$$

where we have,

$$\begin{aligned} g_{\mu\nu}^{(0)} &= 2\left(\partial_{(\mu}\mathsf{r}\partial_{\nu)}\mathsf{v} + m_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)}\right), &(4.1.41\text{a}) \\ g_{\mu\nu}^{(1)} &= -2\Big(\kappa_{(\boldsymbol{\ell})}\partial_{(\mu}\mathsf{v}\partial_{\nu)}\mathsf{v} + 2\pi^{(0)}m_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} + 2\bar{\pi}^{(0)}\bar{m}_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} + 2\mu^{(0)}m_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)} \\ &\quad + \lambda^{(0)}m_{(\mu}^{(0)}m_{\nu)}^{(0)} + \bar{\lambda}^{(0)}\bar{m}_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)}\Big), &(4.1.41\text{b}) \\ g_{\mu\nu}^{(2)} &= 2\Big(2\left(-\mathrm{Re}\,\Psi_2^{(0)} + \left|\pi^{(0)}\right|^2\right)\partial_{(\mu}\mathsf{v}\partial_{\nu)}\mathsf{v} + 2\left(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} - \Psi_3^{(0)}\right)m_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} \\ &\quad + 2\left(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} - \bar{\Psi}_3^{(0)}\right)\bar{m}_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} + 2\left(\left(\mu^{(0)}\right)^2 + \left|\pi^{(0)}\right|^2\right)m_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)} \\ &\quad + \left(2\mu^{(0)}\lambda^{(0)} - \Psi_4^{(0)}\right)m_{(\mu}^{(0)}m_{\nu)}^{(0)} + \left(2\mu^{(0)}\bar{\lambda}^{(0)} - \bar{\Psi}_4^{(0)}\right)\bar{m}_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)}\Big). &(4.1.41\text{c})\end{aligned}$$

4.1.3 Electromagnetism

Going beyond vacuum spacetimes, we expand the Maxwell field scalars radially and iteratively compute them. Beginning with the intrinsic geometry, we realise that $\tau = \nu = \gamma = 0$ from Eq. (4.1.5), $\mu = \bar{\mu}$, $\pi = \alpha + \bar{\beta}$ from Eq. (4.1.8), and the extended gauge choice $\epsilon = \bar{\epsilon}$ from Eq. (4.1.11), which are not affected by the inclusion of electromagnetic fields.

Intrinsic Geometry

On the horizon, we realise $\kappa^{(0)} = \rho^{(0)} = 0$ from Eq. (4.1.9) as previously mentioned. Employing the Raychaudhuri's equation in Eq. (1.2.16), we have, $\sigma^{(0)} = 0$ as previously in Eq. (4.1.10), and additionally,

$$\phi_0^{(0)} = 0, \tag{4.1.42a}$$

which additionally, constraints the Ricci scalars,

$$\Phi_{00}^{(0)} = \Phi_{01}^{(0)} = \Phi_{02}^{(0)} = \Phi_{10}^{(0)} = \Phi_{20}^{(0)} = 0. \tag{4.1.42b}$$

Since the trace of the Maxwell stress energy tensor $\boldsymbol{T}$ vanishes, we have $\Lambda = 0$ everywhere.

In terms of the directional derivatives, we have the non-radial Maxwell's (source-free) equations in Eq. (2.2.32) simplified as,

$$D\phi_1^{(0)} = 0, \tag{4.1.43a}$$

$$D\phi_2^{(0)} - \bar{\delta}\phi_1^{(0)} = 2\pi^{(0)}\phi_1^{(0)} - \kappa_{(\boldsymbol{\ell})}^{(0)}\phi_2^{(0)}, \tag{4.1.43b}$$

such that $\phi_1^{(0)}$ is time-independent for the source-free case.

$\phi_1^{(0)}$ constitutes the initial data that has to be specified on the cross-section $\mathscr{N}_0$, due to its time independence. We further have $\mu^{(0)}$ and $\lambda^{(0)}$ as the transversal expansion and shear, which we specify on $\mathscr{N}_0$, as previously considered.

Further, the two-dimensional Ricci scalar, related to the Gaussian curvature, in Eq. (4.1.22) is modified as,

$${}^{(2)}R = -4\,\mathrm{Re}\,\Psi_2^{(0)} + 8\left|\phi_1^{(0)}\right|^2. \tag{4.1.44}$$

As previously mentioned, we have $\Psi_0^{(0)} = \Psi_1^{(0)} = 0$, and $\Psi_2^{(0)}$ determining the intrinsic horizon geometry. The constraint equation for the Weyl scalar, $\Psi_3^{(0)}$ in Eq. (4.1.23), appropriately changed due to the Ricci components of the tetrad,

$$\Psi_3^{(0)} = \bar{\eth}\mu^{(0)} - \eth\lambda^{(0)} + \left(\mu^{(0)} - \lambda^{(0)}\right)\pi^{(0)} + \phi_2^{(0)}\bar{\phi}_1^{(0)} - \phi_1^{(0)}\bar{\phi}_2^{(0)}, \tag{4.1.45}$$

The Bianchi identity, relating the non-radial derivative of $\Psi_4^{(0)}$ in Eq. (2.2.35) at the horizon,

$$\begin{aligned} D\Psi_4^{(0)} - \bar{\delta}\Psi_3^{(0)} - 2\bar{\phi}_1^{(0)}\bar{\delta}\phi_2^{(0)} = &-3\lambda^{(0)}\Psi_2^{(0)} + 2\Psi_3^{(0)}\left(2\pi^{(0)} + \alpha^{(0)}\right) - 2\kappa_{(\boldsymbol{\ell})}^{(0)}\Psi_4^{(0)} \\ &+ 4\alpha^{(0)}\phi_2^{(0)}\bar{\phi}_1^{(0)} - 4\lambda^{(0)}\left|\phi_1^{(0)}\right|^2, \end{aligned} \tag{4.1.46a}$$

which can be simplified as,

$$D\Psi_4^{(0)} + 2\kappa_{(\ell)}\Psi_4^{(0)} = \left(\bar{\eth} + 5\pi^{(0)}\right)\Psi_3^{(0)} + 2\bar{\phi}_1^{(0)}\left(\bar{\eth} + \pi^{(0)}\right)\phi_2^{(0)} \\ - 3\lambda^{(0)}\Psi_2^{(0)} - 4\lambda^{(0)}\left|\phi_1^{(0)}\right|^2. \tag{4.1.46b}$$

Radial Evolution

The radial evolution of the spin-coefficients is modified appropriately using the field equations in Eq. (2.2.34),

$$\pi^{(1)} = -\Delta\pi = \mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} + \Psi_3^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)}, \tag{4.1.47a}$$

$$\beta^{(1)} = -\Delta\beta = \beta^{(0)}\mu^{(0)} + \alpha^{(0)}\bar{\lambda}^{(0)} + 2\phi_1^{(0)}\bar{\phi}_2^{(0)}, \tag{4.1.47b}$$

$$\alpha^{(1)} = -\Delta\alpha = \lambda^{(0)}\beta^{(0)} + \alpha^{(0)}\mu^{(0)} + \Psi_3^{(0)}, \tag{4.1.47c}$$

$$\lambda^{(1)} = -\Delta\lambda = 2\lambda^{(0)}\mu^{(0)} + \Psi_4^{(0)}, \tag{4.1.47d}$$

$$\mu^{(1)} = -\Delta\mu = \left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + 2\left|\phi_2^{(0)}\right|^2, \tag{4.1.47e}$$

$$\rho^{(1)} = -\Delta\rho = \Psi_2^{(0)}, \tag{4.1.47f}$$

$$\sigma^{(1)} = -\Delta\sigma = 0, \tag{4.1.47g}$$

$$\kappa^{(1)} = -\Delta\kappa = 0, \tag{4.1.47h}$$

$$\epsilon^{(1)} = -\Delta\epsilon = \alpha^{(0)}\bar{\pi}^{(0)} + \beta^{(0)}\pi^{(0)} + \Psi_2^{(0)} + 2\left|\phi_1^{(0)}\right|^2. \tag{4.1.47i}$$

The radial evolution of the connection coefficient a, and the surface gravity $\kappa_{(\ell)}$, are,

$$\epsilon^{(1)} + \bar{\epsilon}^{(1)} = \kappa_{(\ell)}^{(1)} = 2\left|\pi^{(0)}\right|^2 + 2\,\mathrm{Re}\,\Psi_2^{(0)} + 4\left|\phi_1^{(0)}\right|^2, \tag{4.1.48a}$$

$$\epsilon^{(1)} - \bar{\epsilon}^{(1)} = \bar{\pi}^{(0)}a^{(0)} - \pi^{(0)}\left(\bar{\alpha}^{(0)} - \beta^{(0)}\right) + 2i\,\mathrm{Im}\,\Psi_2^{(0)}, \tag{4.1.48b}$$

$$a^{(1)} = \mu^{(0)}a^{(0)} - \lambda^{(0)}\bar{a}^{(0)} + \Psi_3^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)}. \tag{4.1.48c}$$

The remaining (source-free) Maxwell's equations correspond to the radial derivatives,

$$\phi_0^{(1)} = -\Delta\phi_0 = \mu\phi_0 - \sigma\phi_2 - \delta\phi_1, \tag{4.1.49a}$$

$$\phi_1^{(1)} = -\Delta\phi_1 = -2\beta\phi_2 + 2\mu\phi_1 - \delta\phi_2. \tag{4.1.49b}$$

Since there is no radial equation for ϕ_2, we specify it as the initial data on the past light cone $\mathscr{H}_0$. The radial expansion in the radial components leads us to the (source-free) relations, evaluated iteratively in first-order,

$$\phi_0^{(1)} = -\Delta\phi_0 = -\delta\phi_1^{(0)}, \tag{4.1.49c}$$

$$\begin{aligned}\phi_1^{(1)} &= -\Delta\phi_2 = -\delta\phi_2^{(0)} + 2\beta^{(0)}\phi_2^{(0)} + 2\mu^{(0)}\phi_1^{(0)} \\ &= \left(-\eth + \pi^{(0)}\right)\phi_2^{(0)} + 2\mu^{(0)}\phi_1^{(0)}.\end{aligned} \tag{4.1.49d}$$

Further, the radial Bianchi identities in Eq. (2.2.35) are given by, in first-order radial evolution, where we shall use the non-radial (source-free) Maxwell's equations,

$$\Psi_0^{(1)} = 0, \tag{4.1.50a}$$

$$\Psi_1^{(1)} = -\delta\Psi_2^{(0)} - 2\bar{\phi}_1^{(0)}\delta\phi_1^{(0)} = -\eth\Psi_2^{(0)} - 2\bar{\phi}_1^{(0)}\delta\phi_1^{(0)}, \tag{4.1.50b}$$

$$\begin{aligned}\Psi_2^{(1)} = &-\left(\eth + \bar{\pi}^{(0)}\right)\Psi_3^{(0)} + 3\mu^{(0)}\Psi_2^{(0)} - 2\bar{\phi}_1^{(0)}\left(\eth + \bar{\pi}^{(0)}\right)\phi_2^{(0)} + 2\bar{\phi}_2^{(0)}\bar{\delta}\phi_1^{(0)} \\ &+ 4\mu^{(0)}\left|\phi_1^{(0)}\right|^2,\end{aligned} \tag{4.1.50c}$$

$$\begin{aligned}\Psi_3^{(1)} = &-\left(\eth + 2\bar{\pi}^{(0)}\right)\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)} + 2\phi_2^{(1)}\bar{\phi}_1^{(0)} + 2\bar{\phi}_2^{(0)}\left(\eth - 2\pi^{(0)}\right)\phi_2^{(0)} \\ &+ 4\pi^{(0)}\left|\phi_2^{(0)}\right|^2 - 4\lambda^{(0)}\phi_1^{(0)}\bar{\phi}_2^{(0)}.\end{aligned} \tag{4.1.50d}$$

Iteratively, we can now evaluate the expressions for the second-order derivatives by applying the radial derivative in Eq. (4.1.47), and substituting the first derivatives and the Weyl tensors, to arrive at,

$$\begin{aligned}\pi^{(2)} = \Delta^2\pi = &\, 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\right)\pi^{(0)} + 4\bar{\pi}^{(0)}\lambda^{(0)}\mu^{(0)} + 5\mu^{(0)}\Psi_3^{(0)} \\ &+ \lambda^{(0)}\bar{\Psi}_3^{(0)} - \eth\Psi_4^{(0)} - \bar{\pi}^{(0)}\Psi_4^{(0)} + 2\left(\mu^{(0)} + 2\lambda^{(0)}\right)\phi_2^{(0)}\bar{\phi}_1^{(0)},\end{aligned} \tag{4.1.51a}$$

$$\begin{aligned}\beta^{(2)} = \Delta^2\beta = &\, 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\right)\beta^{(0)} + 4\mu^{(0)}\bar{\lambda}^{(0)}\alpha^{(0)} + \alpha^{(0)}\bar{\Psi}_4^{(0)} \\ &+ \bar{\lambda}^{(0)}\Psi_3^{(0)} + \left(-\eth + 2\pi^{(0)}\right)\left|\phi_2^{(0)}\right|^2 + 2\mu^{(0)}\bar{\phi}_2^{(0)}\phi_1^{(0)} + 2\bar{\phi}_1^{(0)}\phi_2^{(1)},\end{aligned} \tag{4.1.51b}$$

$$\begin{aligned}\alpha^{(2)} = \Delta^2\alpha = &\, 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\right)\alpha^{(0)} + 4\mu^{(0)}\lambda^{(0)}\beta^{(0)} + \mu^{(0)}\Psi_3^{(0)} - \eth\Psi_4^{(0)} \\ &- 2\bar{\pi}^{(0)}\Psi_4^{(0)} + 4\mu^{(0)}\Psi_3^{(0)} + 2\phi_1^{(1)}\bar{\phi}_1^{(0)} + 2\bar{\phi}_2^{(0)}\left(\eth - 2\pi^{(0)}\right)\phi_2^{(0)} \\ &+ 4\pi^{(0)}\left|\phi_2^{(0)}\right|^2 - 4\lambda^{(0)}\phi_1^{(0)}\bar{\phi}_2^{(0)},\end{aligned} \tag{4.1.51c}$$

$$\lambda^{(2)} = \Delta^2\lambda = 6\lambda^{(0)}\left(\mu^{(0)}\right)^2 + 2\lambda^{(0)}\left|\lambda^{(0)}\right|^2 + 2\mu^{(0)}\Psi_4^{(0)} + \Psi_4^{(1)}, \tag{4.1.51d}$$

$$\mu^{(2)} = \Delta^2\mu = 2\left(\mu^{(0)}\right)^3 + 6\mu^{(0)}\left|\lambda^{(0)}\right|^2 + \lambda^{(0)}\bar{\Psi}_4^{(0)} + \bar{\lambda}^{(0)}\Psi_4^{(0)} - \bar{\phi}_2^{(0)}\phi_2^{(1)} - \phi_2^{(0)}\bar{\phi}_2^{(1)}, \tag{4.1.51e}$$

$$\rho^{(2)} = \Delta^2\rho = 4\mu^{(0)}\Psi_2^{(0)} - \eth\Psi_3^{(0)} - \bar{\pi}^{(0)}\Psi_3^{(0)} - 2\bar{\phi}_1^{(0)}\left(\eth + \bar{\pi}\right)\phi_2^{(0)}$$

$$+ 2\bar{\phi}_2^{(0)}\bar{\delta}\phi_1^{(0)} + 4\mu^{(0)}\left|\phi_1^{(0)}\right|^2, \tag{4.1.51f}$$

$$\sigma^{(2)} = \Delta^2\sigma = \bar{\lambda}^{(0)}\Psi_2^{(0)}, \tag{4.1.51g}$$

$$\kappa^{(2)} = \Delta^2\kappa = -\eth\Psi_2^{(0)} - \bar{\pi}^{(0)}\Psi_2^{(0)} - 2\bar{\phi}_1^{(0)}\delta\phi_1^{(0)}, \tag{4.1.51h}$$

$$\begin{aligned}\epsilon^{(2)} = \Delta^2\epsilon = 2\left(\alpha^{(0)} + \beta^{(0)}\right)\mu^{(0)}\bar{\pi}^{(0)} + 2\pi^{(0)}\alpha^{(0)}\bar{\lambda}^{(0)} + 2\bar{\pi}^{(0)}\beta^{(0)}\lambda^{(0)} + \alpha^{(0)}\bar{\Psi}_3^{(0)} \\ + \beta^{(0)}\Psi_3^{(0)} - \bar{\eth}\Psi_3^{(0)} + 3\mu^{(0)}\Psi_2^{(0)} - 2\bar{\phi}_1^{(0)}(\bar{\eth} + \bar{\pi})\phi_2^{(0)} + 2\bar{\phi}_2^{(0)}\bar{\delta}\phi_1^{(0)}.\end{aligned} \tag{4.1.51i}$$

Expansion of the Metric

The first-order frame functions are unaffected by the inclusion of the electromagnetic fields, and enter into the frame functions and the metric in second order.

We apply the radial derivative operator to Eq. (4.1.15), to arrive at,

$$\begin{aligned}\mathsf{U}^{(2)} = \Delta^2\mathsf{U} &= -(\Delta\epsilon + \Delta\bar{\epsilon}) - \pi\Delta\Omega - \Omega\Delta\pi - \bar{\pi}\Delta\bar{\Omega} - \bar{\Omega}\Delta\bar{\pi} \\ &= \epsilon^{(1)} + \bar{\epsilon}^{(1)} + \pi^{(0)}\Omega^{(1)} + \Omega^{(0)}\pi^{(1)} + \bar{\pi}^{(0)}\bar{\Omega}^{(1)} + \bar{\Omega}^{(0)}\bar{\pi}^{(1)} \\ &= 4\left|\pi^{(0)}\right|^2 + 2\,\mathrm{Re}\,\Psi_2^{(0)} + 4\left|\phi_1^{(0)}\right|^2,\end{aligned} \tag{4.1.52a}$$

$$\begin{aligned}\Omega^{(2)} = \Delta^2\Omega &= -\Delta\pi - \mu\Delta\Omega - \Omega\Delta\mu - \bar{\lambda}\Delta\bar{\Omega} - \bar{\Omega}\Delta\bar{\lambda} \\ &= \pi^{(1)} + \mu^{(0)}\Omega^{(1)} + \Omega^{(0)}\mu^{(1)} + \bar{\lambda}^{(0)}\bar{\Omega}^{(1)} + \bar{\Omega}^{(0)}\bar{\lambda}^{(1)} \\ &= 2\mu^{(0)}\bar{\pi}^{(0)} + 2\bar{\lambda}^{(0)}\pi^{(0)} + \bar{\Psi}_3^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)},\end{aligned} \tag{4.1.52b}$$

$$\begin{aligned}\xi^{(2),i} = \Delta^2\xi^i &= -\mu\Delta\xi^i - \xi^i\Delta\mu - \bar{\lambda}\Delta\bar{\xi}^i - \bar{\xi}^i\Delta\bar{\lambda} \\ &= \mu^{(0)}\xi^{(1),i} + \xi^{(0),i}\mu^{(1)} + \bar{\lambda}^{(0)}\bar{\xi}^{(1),i} + \bar{\xi}^{(0),i}\bar{\lambda}^{(1)} \\ &= 2\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\right)\xi^{(0),i} + \left(4\mu^{(0)}\bar{\lambda}^{(0)} + \bar{\Psi}_4^{(0)}\right)\bar{\xi}^{(0),i},\end{aligned} \tag{4.1.52c}$$

$$\begin{aligned}\mathsf{X}^{(2),i} = \Delta^2\mathsf{X}^i &= -\pi\Delta\xi^i - \xi^i\Delta\pi - \bar{\pi}\Delta\bar{\xi}^i - \bar{\xi}^i\Delta\bar{\pi} \\ &= \pi^{(0)}\xi^{(1),i} + \xi^{(0),i}\pi^{(1)} + \bar{\pi}^{(0)}\bar{\xi}^{(1),i} + \bar{\xi}^{(0),i}\bar{\pi}^{(1)} \\ &= 2\mu^{(0)}\left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right) + 2\lambda^{(0)}\bar{\pi}^{(0)}\xi^{(0),i} + 2\bar{\lambda}^{(0)}\pi^{(0)}\bar{\xi}^{(0),i} \\ &\quad + \left(\Psi_3^{(0)} + 2\phi_1^{(0)}\bar{\phi}_2^{(0)}\right)\xi^{(0),i} + \left(\bar{\Psi}_3^{(0)} + 2\phi_2^{(0)}\bar{\phi}_1^{(0)}\right)\bar{\xi}^{(0),i}.\end{aligned} \tag{4.1.52d}$$

The frame functions can be expanded in the radial direction, substituting the first and second-order derivatives from Eq. (4.1.34) and Eq. (4.1.52), as

$$\mathsf{U} = \kappa_{(\boldsymbol{\ell})}\mathsf{r} + \left(2\left|\pi^{(0)}\right|^2 + \mathrm{Re}\,\Psi_2^{(0)} + 2\left|\phi_1^{(0)}\right|^2\right)\mathsf{r}^2 + \dots, \tag{4.1.53a}$$

$$\Omega = \bar{\pi}^{(0)}\mathsf{r} + \left(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} + \frac{1}{2}\bar{\Psi}_3^{(0)} + \phi_2^{(0)}\bar{\phi}_1^{(0)}\right)\mathsf{r}^2 + \dots, \tag{4.1.53b}$$

$$\begin{aligned}\xi^i &= \xi^{(0),i} + \left(\mu^{(0)}\xi^{(0),i} + \bar{\lambda}^{(0)}\bar{\xi}^{(0),i}\right)\mathsf{r} + \left(\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\right)\xi^{(0),i}\right. \\ &\quad \left. + \left(2\mu^{(0)}\bar{\lambda}^{(0)} + \frac{1}{2}\bar{\Psi}_4^{(0)}\right)\bar{\xi}^{(0),i}\right)\mathsf{r}^2 + \dots, \end{aligned} \tag{4.1.53c}$$

$$\begin{aligned}\mathsf{X}^i &= \left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right)\mathsf{r} + \left(\left(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} + \frac{1}{2}\Psi_3^{(0)} + \phi_2^{(0)}\bar{\phi}_1^{(0)}\right)\xi^{(0),i}\right. \\ &\quad \left. + \left(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} + \frac{1}{2}\bar{\Psi}_3^{(0)}\bar{\xi}^{(0),i} + \phi_1^{(0)}\bar{\phi}_2^{(0)}\right)\right)\mathsf{r}^2 + \dots \end{aligned} \tag{4.1.53d}$$

$$= \Omega\bar{\xi}^{(0),i} + \bar{\Omega}\xi^{(0),i} + \dots, \tag{4.1.53e}$$

The contravariant spacetime metric $\boldsymbol{g}$, is expressed in terms of the frame functions,

$$g^{\mathsf{rr}} = 2\left(\mathsf{U} + |\Omega|^2\right) \tag{4.1.54a}$$

$$= 2\kappa_{(\boldsymbol{\ell})}\mathsf{r} + 2\left(3\left|\pi^{(0)}\right|^2 + \operatorname{Re}\Psi_2^{(0)} + 2\left|\phi_1^{(0)}\right|^2\right)\mathsf{r}^2 + \dots, \tag{4.1.54b}$$

$$g^{\mathsf{vr}} = 1, \tag{4.1.54c}$$

$$g^{\mathsf{r}\mathsf{x}^i} = \mathsf{X}^i + \bar{\Omega}\xi^i + \Omega\bar{\xi}^i \tag{4.1.54d}$$

$$\begin{aligned} &= 2\left[\left(\pi^{(0)}\xi^{(0),i} + \bar{\pi}^{(0)}\bar{\xi}^{(0),i}\right)\mathsf{r} + \left(\left(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} + \frac{1}{2}\Psi_3^{(0)} + \phi_2^{(0)}\bar{\phi}_1^{(0)}\right)\xi^{(0),i}\right.\right. \\ &\quad \left.\left. + \left(\mu^{(0)}\bar{\pi}^{(0)} + \lambda^{(0)}\pi^{(0)} + \frac{1}{2}\bar{\Psi}_3^{(0)} + \phi_1^{(0)}\bar{\phi}_2^{(0)}\right)\bar{\xi}^{(0),i}\right)\mathsf{r}^2\right] + \dots, \end{aligned} \tag{4.1.54e}$$

$$g^{\mathsf{x}^i\mathsf{x}^j} = \xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j \tag{4.1.54f}$$

$$\begin{aligned} &= \left(\xi^{(0),i}\bar{\xi}^{(0),j} + \xi^{(0),j}\bar{\xi}^{(0),i}\right) + 2\left(\mu^{(0)}\left(\xi^{(0),i}\bar{\xi}^{(0),j} + \xi^{(0),j}\bar{\xi}^{(0),i}\right)\right. \\ &\quad \left. + \lambda^{(0)}\xi^{(0),i}\xi^{(0),j} + \bar{\lambda}^{(0)}\bar{\xi}^{(0),i}\bar{\xi}^{(0),j}\right)\mathsf{r} + \left(3\left(\left(\mu^{(0)}\right)^2 + \left|\lambda^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\right)\right. \\ &\quad \times\left(\xi^{(0),i}\bar{\xi}^{(0),j} + \xi^{(0),j}\bar{\xi}^{(0),i}\right) + \left(6\mu^{(0)}\lambda^{(0)} + \Psi_4^{(0)}\right)\xi^{(0),i}\xi^{(0),j} \\ &\quad \left. + \left(6\mu^{(0)}\bar{\lambda}^{(0)} + \bar{\Psi}_4^{(0)}\right)\bar{\xi}^{(0),i}\bar{\xi}^{(0),j}\right)\mathsf{r}^2 + \dots, \end{aligned} \tag{4.1.54g}$$

where Eq. (4.1.4) was now expanded using Eq. (4.1.53).

The basis 1-forms dual to the null tetrad are,

$$\begin{aligned}\boldsymbol{\ell} &= d\mathsf{r} - \left(\kappa_{(\boldsymbol{\ell})}\mathsf{r} + \left(\operatorname{Re}\Psi_2^{(0)} + 2\left|\phi_1^{(0)}\right|^2\right)\mathsf{r}^2\right)d\mathsf{v} \\ &\quad - \left(\pi^{(0)}\mathsf{r} + \left(\frac{1}{2}\Psi_3^{(0)} + \phi_2^{(0)}\bar{\phi}_1^{(0)}\right)\mathsf{r}^2\right)\xi_i^{(0)}d\mathsf{x}^i \\ &\quad - \left(\bar{\pi}^{(0)}\mathsf{r} + \left(\frac{1}{2}\bar{\Psi}_3^{(0)} + \phi_1^{(0)}\bar{\phi}_2^{(0)}\right)\mathsf{r}^2\right)\bar{\xi}_i^{(0)}d\mathsf{x}^i + \dots, \end{aligned} \tag{4.1.55a}$$

$$\boldsymbol{n} = -d\mathsf{v}, \tag{4.1.55b}$$

$$\begin{aligned}\boldsymbol{m} = &-\left(\bar{\pi}^{(0)}\mathsf{r} + \left(\frac{1}{2}\Psi_3^{(0)} + \phi_2^{(0)}\bar{\phi}_1^{(0)}\right)\mathsf{r}^2\right)d\mathsf{v} + \left(1 - \mu^{(0)}\mathsf{r}\right)\xi_i^{(0)}d\mathsf{x}^i \\ &-\left(\bar{\lambda}^{(0)}\mathsf{r} + \frac{1}{2}\bar{\Psi}_4^{(0)}\mathsf{r}^2\right)\bar{\xi}_i^{(0)}d\mathsf{x}^i + \dots . \end{aligned} \tag{4.1.55c}$$

We now realize the covariant metric, which can be expanded radially,

$$g_{\mu\nu}^{(0)} = 2\left(\partial_{(\mu}\mathsf{r}\partial_{\nu)}\mathsf{v} + m_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)}\right), \tag{4.1.56a}$$

$$\begin{aligned} g_{\mu\nu}^{(1)} = &-2\Big(\kappa_{(\boldsymbol{\ell})}\partial_{(\mu}\mathsf{v}\partial_{\nu)}\mathsf{v} + 2\pi^{(0)}m_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} + 2\bar{\pi}^{(0)}\bar{m}_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} + 2\mu^{(0)}m_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)} \\ &+ \lambda^{(0)}m_{(\mu}^{(0)}m_{\nu)}^{(0)} + \bar{\lambda}^{(0)}\bar{m}_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)}\Big), \end{aligned} \tag{4.1.56b}$$

$$\begin{aligned} g_{\mu\nu}^{(2)} = &\,2\Big(2\Big(-\mathrm{Re}\,\Psi_2^{(0)} - 2\left|\phi_1^{(0)}\right|^2 + \left|\pi^{(0)}\right|^2\Big)\partial_{(\mu}\mathsf{v}\partial_{\nu)}\mathsf{v} + 2\Big(\mu^{(0)}\pi^{(0)} + \lambda^{(0)}\bar{\pi}^{(0)} - \Psi_3^{(0)} \\ &- 2\phi_2^{(0)}\bar{\phi}_1^{(0)}\Big)m_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} + 2\Big(\mu^{(0)}\bar{\pi}^{(0)} + \bar{\lambda}^{(0)}\pi^{(0)} - \bar{\Psi}_3^{(0)} - 2\phi_1^{(0)}\bar{\phi}_2^{(0)}\Big)\bar{m}_{(\mu}^{(0)}\partial_{\nu)}\mathsf{v} \\ &+ 2\Big(\left(\mu^{(0)}\right)^2 + \left|\pi^{(0)}\right|^2 + \left|\phi_2^{(0)}\right|^2\Big)m_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)} + \Big(2\mu^{(0)}\lambda^{(0)} - \Psi_4^{(0)}\Big)m_{(\mu}^{(0)}m_{\nu)}^{(0)} \\ &+ \Big(2\mu^{(0)}\bar{\lambda}^{(0)} - \bar{\Psi}_4^{(0)}\Big)\bar{m}_{(\mu}^{(0)}\bar{m}_{\nu)}^{(0)}\Big). \end{aligned} \tag{4.1.56c}$$

4.2 Schwarzschild Near Horizon

We shall derive the spherically symmetric metric for the Schwarzschild spacetime as an instructive primer to work on the geometry of the neighborhood of the black hole, and the characteristic initial value formulation.

We have the spacetime near the horizon to have $\tau = \nu = \gamma = 0$ from Eq. (4.1.5), $\mu = \bar{\mu}$, $\pi = \alpha + \bar{\beta}$ from Eq. (4.1.8), and the extended gauge choice $\epsilon = \bar{\epsilon}$ from Eq. (4.1.11). On the horizon, we further realise $\kappa^{(0)} = \rho^{(0)} = \sigma^{(0)} = 0$ from Eq. (4.1.9) and Eq. (4.1.10).

4.2.1 Initial Data, Intrinsic Geometry

We proceed with the trivial choice of initial data given by $\Psi_4^{(0)} = 0$, on the transverse null surface, $\mathscr{H}_0$, and choose the common cross section $\mathscr{N}_0$. Further, to provide initial data of the expansion, $\mu^{(0)}$ and shear, $\lambda^{(0)}$ of $\boldsymbol{n}$, we extend the spherical symmetry to $\mathscr{N}_0$ to have $D\mu^{(0)} = 0$ as an angular function, and $\lambda^{(0)} = 0$.

At the Schwarzschild horizon, we assume the metric is Petrov Type D, as in Sec. 2.3.2 to arrive at,

$$\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0, \tag{4.2.1}$$

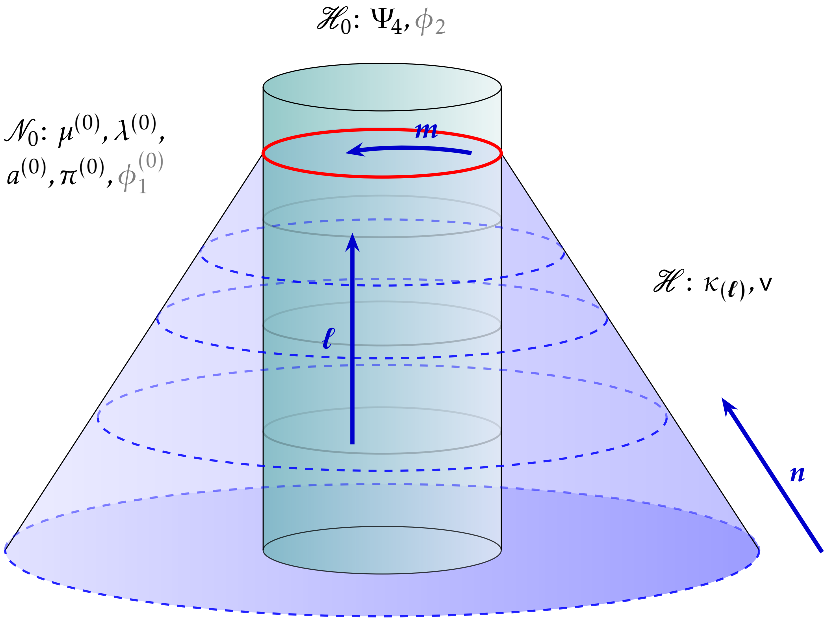

Figure 4.2: The characteristic initial value problem constructing the near-horizon geometry of a spherically symmetric spacetime is presented here. The null surface $\mathscr{H}_0$ represents the horizon whose intrinsic geometry is studied, which is foliated by the temporal coordinate v, defined by the tangent vector $\boldsymbol{\ell}$. The null surface $\mathscr{H}$ is the past null cone, generated by past-directed null geodesics originating from a cross-section $\mathscr{N}_0$, defined using holomorphic coordinates $(\xi, \bar{\xi})$. The affine parameter along the geodesics is r, and the null vector is defined through the directional derivative $\Delta = -\partial_{\mathsf{r}}$, whose foliations are presented as dashed cuts. The spacetime metric is constructed starting with suitable data on $\mathscr{H}_0$, $\mathscr{H}$, and $\mathscr{N}_0$. Additionally, the initial data for the projections of the electromagnetic fields pertaining to the Reissner–Nordström metric is presented.

on the horizon. From Eq. (4.1.21), we have $\Psi_2^{(0)}$ to be an angular function and assume spherical symmetry.

The choice of $\Psi_2^{(0)}$ determines the horizon multipole moments, specifically the mass multipole moments, since we have a non-spinning solution. We realise $\Psi_2^{(0)}$, which is constant, must be completely real, since the angular momentum multipoles, related to $\operatorname{Im}\Psi_2^{(0)}$, vanish. We also have $\mathsf{M} = \frac{1}{2}\mathsf{R}$, from Eq. (3.2.11), and $\kappa_{(\boldsymbol{\ell})} = \frac{1}{2\mathsf{R}}$, from Eq. (3.2.12).

Since the Ricci scalar, which relates the Gaussian curvature, is given on the horizon from Eq. (4.1.22), as ${}^{(2)}R = -4\operatorname{Re}\Psi_2^{(0)}$, we can use the Gauss-Bonnet theorem to have,

$$8\pi = \oint_{\mathcal{N}_0} {}^{(2)}R\,\boldsymbol{\epsilon} = -4\oint_{\mathcal{N}_0} \operatorname{Re}\Psi_2^{(0)}\,\boldsymbol{\epsilon}, \tag{4.2.2a}$$

hence, we use the constancy of $\Psi_2^{(0)}$, and the area of the horizon $\mathsf{A} = 4\pi\mathsf{R}^2$, to arrive at,

$$\Psi_2^{(0)} = \operatorname{Re}\Psi_2^{(0)} = -\frac{1}{2\mathsf{R}^2} = -\frac{1}{8\mathsf{M}^2}, \tag{4.2.2b}$$

thus determining the intrinsic horizon geometry.

Now, we can relate the angular function $\mu^{(0)}$ to $\Psi_2^{(0)}$ by the time evolution field equations in Eq. (4.1.24), to have,

$$\kappa_{(\boldsymbol{\ell})}\mu^{(0)} = \bar{\eth}\pi^{(0)} + \left|\pi^{(0)}\right|^2 + \Psi_2^{(0)}, \tag{4.2.3a}$$

where we have used $\pi^{(0)} = 0$, a direct consequence of the assumed spherical symmetry of the Schwarzschild horizon, which canonically fixes the value,

$$\mu^{(0)} = -\frac{1}{2\mathsf{M}}. \tag{4.2.3b}$$

4.2.2 Radial Evolution

Now, we use the radial evolution relations in Eq. (4.1.16), to have,

$$\Delta\mu = -\frac{d\mu}{d\mathsf{r}} = -\mu^2 - |\lambda|^2, \tag{4.2.4a}$$

$$\Delta\lambda = -\frac{d\lambda}{d\mathsf{r}} = 2\lambda\mu, \tag{4.2.4b}$$

$$\Delta\sigma = -\frac{d\sigma}{d\mathsf{r}} = -\mu\sigma - \bar{\lambda}\varrho, \tag{4.2.4c}$$

where we have used $\Psi_4 = 0$.

We use $\lambda^{(0)} = 0$, $\mu^{(0)} = -\frac{1}{2\mathsf{M}}$, and $\sigma^{(0)} = 0$,

$$\lambda = \sigma = 0, \tag{4.2.5a}$$

$$\mu = -\frac{1}{r}, \tag{4.2.5b}$$

which we have solved sequentially.

Similarly, the radial Bianchi identities in Eq. (4.1.17) can be sequentially solved through,

$$\Delta\Psi_3 = \frac{4}{r}\Psi_3, \tag{4.2.6a}$$
$$\Delta\Psi_2 - \delta\Psi_3 = \frac{3}{r}\Psi_2 + 2\beta\Psi_3, \tag{4.2.6b}$$
$$\Delta\Psi_1 - \delta\Psi_2 = \frac{2}{r}\Psi_1, \tag{4.2.6c}$$
$$\Delta\Psi_0 - \delta\Psi_1 = \frac{1}{r}\Psi_0 - 2\beta\Psi_1, \tag{4.2.6d}$$

where we substitute the initial conditions.

We have $\Psi_3^{(0)} = 0$, and $\Psi_2^{(0)} = -\frac{1}{2R^2}$, to arrive at

$$\Psi_0 = \Psi_1 = \Psi_3 = 0, \tag{4.2.7a}$$
$$\Psi_2 = -\frac{M}{r^3}, \tag{4.2.7b}$$
$$\tag{4.2.7c}$$

which extends the algebraic symmetry of the Weyl tensors.

Now, we can solve the remaining radial evolution equations,

$$\Delta\pi = -\frac{d\pi}{dr} = \frac{1}{r}\pi, \tag{4.2.8a}$$
$$\Delta\kappa = -\frac{d\kappa}{dr} = -\rho\bar{\pi}, \tag{4.2.8b}$$
$$\Delta\rho = -\frac{d\rho}{dr} = \frac{1}{r}\rho + \frac{M}{r^3}, \tag{4.2.8c}$$
$$\Delta\epsilon = -\frac{d\epsilon}{dr} = -\alpha\bar{\pi} - \beta\pi + \frac{M}{r^3}, \tag{4.2.8d}$$
$$\Delta a = -\frac{da}{dr} = \frac{1}{r}a, \tag{4.2.8e}$$

Thereby, we use $\pi^{(0)} = 0$, $\epsilon^{(0)} = \frac{1}{2}\kappa_{(\ell)} = \frac{1}{8M}$, $\rho^{(0)} = 0$,

$$\pi = \kappa = 0, \tag{4.2.9a}$$
$$\rho = -\frac{1}{2r}\left(1 - \frac{2M}{r}\right), \quad \epsilon = \frac{M}{2r^2}, \tag{4.2.9b}$$
$$a = \frac{a^{(0)}}{r}. \tag{4.2.9c}$$

4.2.3 Frame Functions, Spacetime Metric

Further, we can integrate the radial frame function evolution, to arrive at, by substituting the spin coefficients in Eq. (4.1.15),

$$\Delta \mathrm{U} = -\frac{d\mathrm{U}}{d\mathrm{r}} = \frac{\mathrm{M}}{\mathrm{r}^2}, \tag{4.2.10a}$$

$$\Delta \Omega = -\frac{d\Omega}{d\mathrm{r}} = -\frac{1}{\mathrm{r}}\Omega, \tag{4.2.10b}$$

$$\Delta \xi^i = -\frac{d\xi^i}{d\mathrm{r}} = -\frac{1}{\mathrm{r}}\xi^i, \tag{4.2.10c}$$

$$\Delta \mathrm{X}^i = -\frac{d\mathrm{X}^i}{d\mathrm{r}} = 0. \tag{4.2.10d}$$

Here we use the initial conditions, $\mathrm{U}^{(0)} = \mathrm{X}^{(0)} = \Omega^{(0)} = 0$ on the horizon, and $\xi^{(0),i}$ is determined by the holomorphic coordinates, to have

$$\Omega = \mathrm{X}^i = 0, \tag{4.2.11a}$$

$$\mathrm{U} = \frac{2\mathrm{M} - \mathrm{r}}{2\mathrm{r}}, \tag{4.2.11b}$$

$$\xi^i = \xi^{(0),i}\mathrm{r}, \tag{4.2.11c}$$

The reconstructed null tetrad, through Eq. (4.1.3), is,

$$\ell^i \nabla_i = D = \partial_{\mathrm{v}} - \left(1 - \frac{2\mathrm{M}}{\mathrm{r}}\right)\partial_{\mathrm{r}}, \tag{4.2.12a}$$

$$n^i \nabla_i = \Delta = -\partial_{\mathrm{r}}, \tag{4.2.12b}$$

$$m^i \nabla_i = \delta = \mathrm{r}\xi^{(0),i}\partial_{\mathrm{x}^i}, \tag{4.2.12c}$$

Furthermore, the metric components are evaluated from Eq. (4.1.38),

$$g^{\mathrm{rr}} = 2\mathrm{U} = \frac{2\mathrm{M}}{\mathrm{r}} - 1, \tag{4.2.13a}$$

$$g^{\mathrm{vr}} = 1, \quad g^{\mathrm{rx}^i} = 0, \tag{4.2.13b}$$

$$g^{\mathrm{x}^i\mathrm{x}^j} = \xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j = 2\,\mathrm{Re}\{\xi^{(0),i}\bar{\xi}^{(0),j}\}\mathrm{r}^2. \tag{4.2.13c}$$

The basis 1-forms dual to the null tetrad in Eq. (4.2.12) are thereby,

$$\boldsymbol{\ell} = -\frac{1}{2}\left(1 - \frac{2\mathrm{M}}{\mathrm{r}}\right)d\mathrm{v} + d\mathrm{r}, \tag{4.2.14a}$$

$$\boldsymbol{n} = -d\mathrm{v}, \quad \boldsymbol{m} = \frac{1}{2}\mathrm{r}\xi^{(0)}_i d\mathrm{x}^i. \tag{4.2.14b}$$

Hence, we have the covariant metric, given by the basis one-forms $g_{\mu\nu} = 2\left(-\ell_{(\mu}n_{\nu)} + m_{(\mu}\bar{m}_{\nu)}\right)$, in the gauge choices we have used,

$$ds^2 = -\left(1 - \frac{2\mathrm{M}}{\mathrm{r}}\right)d\mathrm{v}^2 + 2d\mathrm{v}d\mathrm{r} + \frac{1}{2\mathrm{r}^2}\,\mathrm{Re}\{\xi_i^{(0)}\bar{\xi}_j^{(0)}\}d\mathrm{x}^i d\mathrm{x}^j \tag{4.2.15}$$

This concludes the classical IH formalism for the spherically symmetric Schwarzschild solution, with the Hamiltonian calculations giving us appropriate values for mass and surface gravity, geometric constraints on the IH needed to be satisfied in accordance with the algebraic properties of the Weyl tensor, the multipole moments yielded Ψ_2, and the radial and angular field equations accomplished the rest.

4.3 Reissner-Nordström Near Horizon

Beyond spherically symmetric vacuum spacetimes, we shall consider the extension to Maxwell's field.

Due to the pertaining spherical symmetry, we adapt the near-horizon tetrad as considered earlier. Using the same gauge as the Schwarzschild metric, we have $\tau = \nu = \gamma = 0$, $\epsilon = \bar{\epsilon}$, $\mu = \bar{\mu}$, and $\pi = \alpha + \bar{\beta}$, from Eq. (4.1.5), Eq. (4.1.10), and Eq. (4.1.11).

4.3.1 Initial Data, Intrinsic Geometry

We proceed with choosing a transverse null surface, $\mathscr{H}_0$, and the common cross-section, with the trivial choice of initial data given by $\Psi_4^{(0)} = 0$, $\phi_2^{(0)} = 0$, $D\mu^{(0)} = 0$, and $\lambda^{(0)} = 0$ respecting spherical symmetry.

We choose the metric of the horizon to be Petrov Type D, to have $\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0$, and the choice of $\Psi_2^{(0)}$, $\phi_1^{(0)}$ determines the horizon multipole moments. From Raychaudhari's equation, we have $\phi_0^{(0)} = 0$ in Eq, (4.1.42).

We impose spherical symmetry on the Maxwell scalars on the horizon, such that we impose the physical constraint, $\phi_2 = 0$, which satisfies the non-radial (source-free) Maxwell equations. The quantity $\phi_1^{(0)}$ is time-independent, and assumed to be constant on $\mathscr{N}_0$, such that its real and imaginary parts describe the electric and magnetic flux density through $\mathscr{N}_0$, respectively.

Due to spherical symmetry, we require that the angular momentum of the horizon in Eq. (3.2.15) must vanish, thereby resulting in,

$$\mathrm{Im}\,\Psi_2^{(0)} = 0, \tag{4.3.1}$$

due to its relation to the curl of the rotation 1-form.

Hence, the horizon mass in Eq. (3.2.18) is $\mathsf{M} = \frac{1}{2\mathsf{R}}\left(\mathsf{R}^2 + \mathsf{q}^2\right)$ and the surface gravity simplifies as $\kappa_{(\boldsymbol{\ell})} = \frac{\mathsf{R}^4 - \mathsf{q}^4}{2\mathsf{R}^3(\mathsf{R}^2+\mathsf{q}^2)} = \frac{1}{2\mathsf{R}} - \frac{\mathsf{q}^2}{2\mathsf{R}^3}$ from Eq. (3.2.19). We note the charge of the horizon from the definition in Eq. (3.2.17),

$$\mathsf{Q} + i\mathsf{P} \doteq \frac{1}{2\pi}\oint_{\mathcal{N}_0} \phi_1^{(0)}\, \boldsymbol{\epsilon} = 2\mathsf{R}^2\phi_1^{(0)}, \tag{4.3.2a}$$

by spherical symmetry for $\phi_1^{(0)}$. Thereby, the effective charge of the horizon is,

$$\mathsf{q}^2 = \mathsf{Q}^2 + \mathsf{P}^2 = 4\mathsf{R}^4\left|\phi_1^{(0)}\right|^2. \tag{4.3.2b}$$

Further, $\Psi_2^{(0)}$ is time-independent and completely real, and we determine it from the Gauss-Bonnet Theorem, using the 2-dimensional Ricci scalar in Eq. (4.1.44),

$$8\pi = \oint_{\mathcal{N}_0} {}^{(2)}R\, \boldsymbol{\epsilon} = \oint_{\mathcal{N}_0}\left(-4\,\mathrm{Re}\,\Psi_2^{(0)} + 8\left|\phi_1^{(0)}\right|^2\right)\boldsymbol{\epsilon}, \tag{4.3.3}$$

hence, we have, on the horizon,

$$\Psi_2^{(0)} = \mathrm{Re}\,\Psi_2^{(0)} = -\frac{1}{2\mathsf{R}^2} + \frac{\mathsf{q}^2}{2\mathsf{R}^4} \tag{4.3.4}$$

Using the form of $\Psi_2^{(0)}$ at the horizon, we can arrive at the transversal expansion, which we assume is an angular function, to have $\kappa_{(\boldsymbol{\ell})}\mu^{(0)} = \Psi_2^{(0)}$ to arrive at, $\mu^{(0)} = -\frac{1}{\mathsf{R}}$, which is similar to the case of the Schwarzschild tetrad.

4.3.2 Radial Evolution

Now, utilising the radial evolution relations in Eq. (4.1.47), we have,

$$\Delta\mu = -\frac{d\mu}{d\mathsf{r}} = -\mu^2 - |\lambda|^2, \tag{4.3.5a}$$

$$\Delta\lambda = -\frac{d\lambda}{d\mathsf{r}} = 2\lambda\mu, \tag{4.3.5b}$$

$$\Delta\sigma = -\frac{d\sigma}{d\mathsf{r}} = -\mu\sigma - \bar{\lambda}\rho. \tag{4.3.5c}$$

Sequentially, we solve by using the spherically symmetric data on the horizon, $\lambda^{(0)} = 0$, $\mu^{(0)} = -\frac{1}{\mathsf{R}}$, $\sigma^{(0)} = 0$,

$$\lambda = \sigma = 0, \tag{4.3.6a}$$

$$\mu = -\frac{1}{\mathsf{r}}. \tag{4.3.6b}$$

The (source-free) Maxwell's equations in Eq. (4.1.49) correspond to the radial derivatives,

$$\phi_0^{(1)} = -\frac{d\phi_0}{d\mathrm{r}} = \frac{1}{\mathrm{r}}\phi_0, \tag{4.3.7a}$$

$$\phi_1^{(1)} = -\frac{d\phi_1}{d\mathrm{r}} = \frac{2}{\mathrm{r}}\phi_1, \tag{4.3.7b}$$

We use the initial conditions of $\phi_0^{(0)} = 0$, and $\left|\phi_1^{(0)}\right|^2 = \frac{\mathrm{q}^2}{4\mathrm{R}^4}$ such that, we have the iteratively,

$$\phi_0 = 0, \tag{4.3.8a}$$

$$\phi_1 = \frac{\mathrm{q}}{2\mathrm{r}^2}\exp(i\psi), \tag{4.3.8b}$$

and we have the physical constraint condition $\phi_2 = 0$.

The radial Bianchi identities in Eq. (4.1.50) simplify by using the spin coefficients,

$$\Delta\Psi_3 = -\frac{\partial\Psi_3}{\partial\mathrm{r}} = \frac{4}{\mathrm{r}}\Psi_3, \tag{4.3.9a}$$

$$\Delta\Psi_2 = -\frac{\partial\Psi_2}{\partial\mathrm{r}} = \frac{3}{\mathrm{r}}\Psi_2 + 2\beta\Psi_3 + \delta\Psi_3 - \frac{\mathrm{q}^2}{\mathrm{r}^5}, \tag{4.3.9b}$$

$$\Delta\Psi_1 = -\frac{\partial\Psi_1}{\partial\mathrm{r}} = \frac{2}{\mathrm{r}}\Psi_1 + \delta\Psi_2, \tag{4.3.9c}$$

$$\Delta\Psi_0 = -\frac{\partial\Psi_0}{\partial\mathrm{r}} = \frac{1}{\mathrm{r}}\Psi_0 - 2\beta\Psi_1 + \delta\Psi_1. \tag{4.3.9d}$$

Solving sequentially, we have by substituting $\Psi_0^{(0)} = \Psi_1^{(0)} = \Psi_3^{(0)} = \Psi_4^{(0)} = 0$, $\Psi_2^{(0)} = -\frac{1}{2\mathrm{R}^2} + \frac{\mathrm{q}^2}{2\mathrm{R}^4}$,

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \tag{4.3.10a}$$

$$\Psi_2 = -\frac{\mathrm{M}}{\mathrm{r}^3} + \frac{\mathrm{q}^2}{\mathrm{r}^4}. \tag{4.3.10b}$$

Now, we encounter the other radial evolution equations for the spin coefficients in Eq. (4.1.47),

$$\Delta\pi = -\frac{d\pi}{d\mathrm{r}} = \frac{1}{\mathrm{r}}\pi, \tag{4.3.11a}$$

$$\Delta\kappa = -\frac{d\kappa}{d\mathrm{r}} = -\varrho\bar{\pi}, \tag{4.3.11b}$$

$$\Delta\varrho = -\frac{d\varrho}{d\mathrm{r}} = \frac{1}{\mathrm{r}}\varrho + \frac{\mathrm{M}}{\mathrm{r}^3} - \frac{\mathrm{q}^2}{\mathrm{r}^4}, \tag{4.3.11c}$$

$$\Delta\epsilon = -\frac{d\epsilon}{d\mathrm{r}} = -\alpha\bar{\pi} - \beta\pi + \frac{\mathrm{M}}{\mathrm{r}^3} - \frac{\mathrm{q}^2}{2\mathrm{r}^4}, \tag{4.3.11d}$$

$$\Delta a = -\frac{da}{d\mathrm{r}} = \frac{1}{\mathrm{r}}a, \tag{4.3.11e}$$

Thereby, we use $\pi^{(0)} = 0$, $\epsilon^{(0)} = \frac{1}{2}\kappa_{(\boldsymbol{\ell})} = -\frac{1}{8\mathrm{M}} - \frac{1}{2\mathrm{MR}}$, $\rho^{(0)} = 0$,

$$\pi = \kappa = 0, \tag{4.3.12a}$$

$$\rho = -\frac{1}{2\mathrm{r}}\left(1 - \frac{2\mathrm{M}}{\mathrm{r}} + \frac{\mathrm{q}^2}{\mathrm{r}^2}\right), \tag{4.3.12b}$$

$$\epsilon = \frac{\mathrm{M}}{2\mathrm{r}^2} - \frac{\mathrm{q}^2}{2\mathrm{r}^3}, \quad a = \frac{a^{(0)}}{\mathrm{r}}. \tag{4.3.12c}$$

4.3.3 Frame Functions, Spacetime Metric

Substituting the spin coefficients in Eq. (4.1.15), we derive expressions similar to those previously encountered. The differences include,

$$\Delta\mathrm{U} = -\frac{d\mathrm{U}}{d\mathrm{r}} = \frac{\mathrm{M}}{\mathrm{r}^2} - \frac{\mathrm{q}^2}{\mathrm{r}^3}, \tag{4.3.13a}$$

Using the initial data on the horizon, $\mathrm{U}^{(0)} = 0$, we have,

$$\mathrm{U} = \frac{2\mathrm{Mr} - \mathrm{q}^2 - \mathrm{r}^2}{2\mathrm{r}^2}. \tag{4.3.13b}$$

The reconstructed null tetrad, from the frame functions, through Eq. (4.1.3), is,

$$\ell^i\nabla_i = D = \partial_\mathrm{v} - \left(1 - \frac{2\mathrm{M}}{\mathrm{r}} + \frac{\mathrm{q}^2}{\mathrm{r}^2}\right)\partial_\mathrm{r}, \tag{4.3.14a}$$

$$n^i\nabla_i = \Delta = -\partial_\mathrm{r}, \quad m^i\nabla_i = \delta = \mathrm{r}\xi^{(0),i}\partial_{\mathrm{x}^i}, \tag{4.3.14b}$$

The metric components are evaluated from Eq. (4.1.54), differ in the $g^{\mathrm{rr}} = \frac{2\mathrm{M}}{\mathrm{r}} - \frac{\mathrm{q}^2}{\mathrm{r}^2} - 1$, such that the co-tetrad dual to Eq. (4.3.14) is given by,

$$\boldsymbol{\ell} = -\frac{1}{2}\left(1 - \frac{2\mathrm{M}}{\mathrm{r}} + \frac{\mathrm{q}^2}{\mathrm{r}^2}\right)d\mathrm{v} + d\mathrm{r}, \tag{4.3.15a}$$

$$\boldsymbol{n} = -d\mathrm{v}, \quad \boldsymbol{m} = \frac{1}{2}\mathrm{r}\xi_i^{(0)}d\mathrm{x}^i, \tag{4.3.15b}$$

where we have used the dual, $\xi_i^{(0)}$.

Thereby, the covariant metric in our gauge is,

$$ds^2 = -\left(1 - \frac{2\mathrm{M}}{\mathrm{r}} + \frac{\mathrm{q}^2}{\mathrm{r}^2}\right)d\mathrm{v}^2 + 2d\mathrm{v}d\mathrm{r} + \frac{1}{2\mathrm{r}^2}\mathrm{Re}\{\xi_i^{(0)}\bar{\xi}_j^{(0)}\}d\mathrm{x}^i d\mathrm{x}^j \tag{4.3.16}$$

Chapter 5

Tidal Deformabilities

The response of any self-gravitating object under the influence of an external tidal field can be divided into the conservative and dissipative components. The conservative part encodes the information about the tidal deformation of the body, while the dissipative part describes the absorption of the emitted gravitational waves by the deformed object.

The study of weakly deformed black holes has a rich history, beginning with the investigations of the Schwarzschild geometry and later extended to the Kerr background. Consequently, wave equations for linearized massless particles are now known for all black hole backgrounds in a form that can be separated into ordinary differential equations. These equations allow us to characterize how a black hole responds to small perturbations.

We will explore a related aspect of the linear response of a black hole to a static external field. Specifically, we will examine the response of a black hole to background profiles of gravitational tidal perturbations that maintain the invariance of the cross-section of a black hole's horizon. At leading order, tidally perturbed black hole horizons can be modeled as isolated horizons, and the non-vanishing fluxes of infalling radiation can be calculated at linear order using perturbation theory. The response of an object to a long-wavelength tidal gravitational field is encapsulated in the *Love numbers*, which can be regarded as measures of the object's deformability or rigidity.

5.1 Love Numbers

Consider a black hole in a tidal environment, which can be created by an external gravitational field generated by other objects, or by an external scalar or vector field. When influenced by a weak static tidal field, black holes exhibit vanishing gravitational Love numbers.

Love numbers quantify how an object responds to tidal forces: if the external tidal field behaves like r^ℓ at a large distance r, where ℓ represents the relevant multipole moment, an object like a star that experiences tidal deformation will generate a response field that decays as $1/r^{\ell+1}$ at large distances. The coefficient of this decay, normalized by the amplitude of the tidal field, is referred to as a Love number. There can be multiple Love numbers (or even Love tensors), as different types of tidal fields may be applied.

For a weak static tidal field, the problem reduces to studying static linear perturbations around the object of interest, whether it be a star or a black hole. At a large distance from the object, the perturbation satisfies the Laplace equation, which admits two linearly independent solutions at large r, one with a growing r^ℓ profile and another with a decaying $1/\mathsf{r}^{\ell+1}$ profile at multipole order ℓ.

For a star, the growing (tidal) solution is generally accompanied by the decaying tail, signifying deformability. In contrast, for a black hole, the growing solution is regular at the horizon, while the decaying solution is not and thus is discarded. These two facts together indicate that the Love numbers for a black hole vanish.

5.1.1 Newtonian Theory

Consider an isolated and spherical Newtonian body of mass M and equilibrium radius R, as well as a Cartesian coordinate system $(\mathsf{t}, \vec{\mathsf{x}})$. We embed this body in an external gravitational potential, $\Phi_{\rm ext}(\mathsf{t}, \vec{\mathsf{x}})$. Performing a Taylor series expansion of this external potential about the centre of mass $\vec{\mathsf{x}}_{\rm COM}$, the tidal environment of the body can be characterized for an arbitrary multipolar index ℓ by a set of time-dependent, symmetric, and trace-free (STF) tidal moments,

$$\mathcal{E}_L(\mathsf{t}) = -\frac{1}{(\ell-2)!}\partial_{\langle L\rangle}\Phi_{\rm ext}(\mathsf{t}, \vec{\mathsf{x}}_{\rm COM}), \tag{5.1.1}$$

where $L = i_1 \cdots i_\ell$ is a multi-index, while the brackets denote the STF part, and appropriate normalization is used for generalizing to the relativistic case.

Accordingly, the perturbed body develops a contribution $\Phi_{\rm body}(\mathsf{t}, \vec{\mathsf{x}})$ to the total gravitational potential, which can be characterized by a set of time-dependent, STF multipole moments,

$$I_L(\mathsf{t}) = \int_{\mathbb{R}^3} d^3\mathsf{x}\ \delta\rho(\mathsf{t}, \vec{\mathsf{x}})\mathsf{x}^{\langle L\rangle}, \tag{5.1.2}$$

where $\delta\rho(\mathsf{t}, \vec{\mathsf{x}})$ is the mass density perturbation induced by the applied tidal field.

In the centre of mass frame, the dipole moment vanishes, thus the total gravitational potential $\Phi = \Phi_{\rm ext} + \Phi_{\rm body}$ is

$$\Phi(\mathsf{t}, \vec{\mathsf{x}}) = -\sum_{\ell=2}^{\infty}\frac{1}{\ell(\ell-1)}\mathsf{x}^L\mathcal{E}_L(\mathsf{t}) + \frac{\mathsf{M}}{\mathsf{r}} + \sum_{\ell=2}^{\infty}\frac{(2\ell-1)!!}{\ell!}\frac{\mathsf{n}^L I_L(t)}{\mathsf{r}^{\ell+1}}, \tag{5.1.3}$$

with $\mathsf{r} = \left|\vec{\mathsf{x}}\right|$, and $\vec{\mathsf{n}} = \frac{1}{\mathsf{r}}\vec{\mathsf{x}}$ is the unit vector in the x direction. For a given value of the multipolar index ℓ, the first term scales as r^ℓ and corresponds to the applied multipole tidal field $\mathcal{E}_L(\mathrm{t})$, while the third term scales as $1/\mathrm{r}^{\ell+1}$ and corresponds to the induced multipole moment $I_L(\mathrm{t})$ of the body.

Let $\mathcal{R}$ denote the typical length scale of variation of the tidal environment. The tidal field typically scales as $\frac{1}{\mathcal{R}^\ell}\Phi_{\mathrm{ext}}$. Consequently, the multipolar expansion of the external potential Φ_{ext}, is only valid in the domain $0 \leq \mathrm{r} \leq \mathcal{R}$. On the other hand, the multipolar expansion of the body's response is valid for all $\mathrm{r} \geq \mathrm{R}$. Therefore, the form of the total gravitational potential is valid in the region $\mathrm{R} \leq \mathrm{r} \leq \mathcal{R}$. This separation of scales ensures that the tidal field remains quasi-static, with the timescales associated with changes in the tidal field being adiabatic compared to the internal timescale of the body.

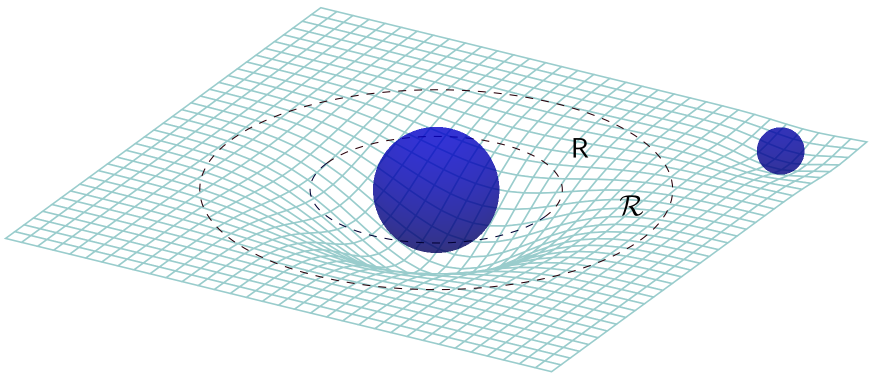

Figure 5.1: Representation of the region of the spacetime where the expansions of the potential of the tidal deformability are valid. We have an isolated and spherical Newtonian body of mass M and equilibrium radius R. $\mathcal{R}$ denotes the typical length scale of variation of the tidal environment. The form of the total gravitational potential is valid in the region $\mathrm{R} \leq \mathrm{r} \leq \mathcal{R}$.

For a weak and slowly varying external tidal field (adiabatic approximation), the induced mass multipole moments are proportional to the applied tidal multipoles, such that,

$$I_L(\mathrm{t}) = -\lambda_\ell \mathcal{E}_L(\mathrm{t}), \tag{5.1.4}$$

where the λ_ℓ is the tidal deformability parameter, a real-valued constant that depends only on the internal structure of the body, and scales as $\mathrm{R}^{2\ell+1}$. We shall use the dimensionless tidal love numbers, defined by the rescaling,

$$\lambda_\ell = \frac{2(\ell-2)!}{(2\ell-1)!!} k_\ell \mathrm{R}^{2\ell+1}, \tag{5.1.5}$$

such that the Newtonian gravitational potential in Eq. (5.1.3) then becomes,

$$\Phi(\mathrm{t},\mathrm{r},\xi,\bar{\xi}) = \frac{\mathrm{M}}{\mathrm{r}} - \sum_{\ell,m} \frac{1}{\ell(\ell-1)} \mathrm{r}^\ell \left[1 + 2k_\ell \left(\frac{\mathrm{R}}{\mathrm{r}}\right)^{2\ell+1}\right] \mathcal{E}_{\ell,m}(\mathrm{t}) Y_{\ell,m}(\xi,\bar{\xi}), \tag{5.1.6}$$

where we have conveniently introduced stereographic coordinates, and expanded the time-dependent scalar field $\mathsf{n}^L \mathcal{E}_L(t)$ over the orthonormal basis of spherical harmonics.

In the above expression, the Newtonian potential of the black hole itself with mass M can be identified, as well as the external potential, proportional to k_ℓ denotes the (gravitational) multipole moment that is induced in the black hole. Thereby, gravitational Love numbers for a specific multipole are the dimensionless ratio of the multipolar component of the body's induced gravitational potential to the multipolar component of the external gravitational field.

5.1.2 Relativistic Love Numbers

In the Newtonian limit, the components of the metric with respect to the Cartesian coordinates are given in terms of the gravitational potential as the effective frame functions,

$$g_{00} = -1 + \frac{2\Phi}{c^2} + \mathcal{O}(c^{-4}), \tag{5.1.7a}$$

$$g_{0\mathrm{i}} = \mathcal{O}(c^{-3}), \tag{5.1.7b}$$

$$g_{\mathrm{ij}} = \delta_{\mathrm{ij}}\left(1 + \frac{2\Phi}{c^2}\right) + \mathcal{O}(c^{-4}). \tag{5.1.7c}$$

The Newtonian limit of the Weyl scalar projection Ψ_0 in Eq. (2.2.21) is considered at the flat spacetime limit of the null vectors $\boldsymbol{\ell}$ and $\boldsymbol{m}$, by using the spin-operators in Eq. (2.4.23) to have,

$$\begin{aligned}
\lim_{c\to\infty} c^2\Psi_0(\mathrm{t},\mathrm{r},\xi,\bar{\xi}) &= -\frac{1}{\mathrm{r}^2}\sum_{\ell,m} \eth^2\Phi(\mathrm{t},\mathrm{r},\xi,\bar{\xi}) \\
&= \sum_{\ell,m} \frac{1}{\ell(\ell-1)}\mathrm{r}^{\ell-2}\left[1 + 2k_\ell\left(\frac{\mathrm{R}}{\mathrm{r}}\right)^{2\ell+1}\right]\mathcal{E}_{\ell,m}(\mathrm{t})\eth^2 Y_{\ell,m}(\xi,\bar{\xi}) \\
&= \sum_{\ell,m} \mathrm{r}^{\ell-2}\left[1 + 2k_\ell\left(\frac{\mathrm{R}}{\mathrm{r}}\right)^{2\ell+1}\right]\alpha_{\ell,m}(\mathrm{t})\,{}_2Y_{\ell,m}(\xi,\bar{\xi}),
\end{aligned} \tag{5.1.8}$$

where we have defined the coefficients $\alpha_{\ell,m}(\mathrm{t}) = \sqrt{\frac{(\ell+2)(\ell+1)}{\ell(\ell-1)}}\mathcal{E}_{\ell,m}(\mathrm{t})$, by using $\eth^2 Y_{\ell,m} = \sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\,{}_2Y_{\ell,m} = \sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}\,{}_2Y_{\ell,m}$ from Eq. (2.4.23).

Similarly, for the radiative component of the Weyl tensor given by Ψ_4, we consider the flat spacetime limit of the null vectors $\boldsymbol{n}$ and $\bar{\boldsymbol{m}}$, to have

$$\lim_{c\to\infty} c^2\Psi_4(\mathrm{t},\mathrm{r},\xi,\bar{\xi}) = \frac{1}{4}\sum_{\ell,m} \mathrm{r}^{\ell-2}\left[1 + 2k_\ell\left(\frac{\mathrm{R}}{\mathrm{r}}\right)^{2\ell+1}\right]\alpha_{\ell,m}(\mathrm{t})\,{}_{-2}Y_{\ell,m}(\xi,\bar{\xi}), \tag{5.1.9}$$

where we realize $\bar{\eth}^2 Y_{\ell,m} = \sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}\,{}_{-2}Y_{\ell,m}$ from Eq. (2.4.23).

The Weyl scalars Ψ_0 and Ψ_4 in the Newtonian limit provide the way forward for defining a gauge-invariant tidal response function. Although the definition of the tidal response function in terms of the Weyl scalar is relativistically invariant, it is dependent on the spacetime foliation through the choice of the tetrad vectors.

5.1.3 Surficial Love Numbers

Analogous to gravitational Love numbers, we can define surficial Love numbers, h_ℓ as the deformation of the body's surface, also expanded in multipole moments. We measure the tidal deformation, as the implicit change in the radius of the (spherically symmetric) body from $\mathsf{r} = \mathsf{R}$ to a new equilibrium radius, $\mathsf{r} = \mathsf{R} + \hat{\mathsf{R}}$, such that

$$\hat{\mathsf{R}}(\mathsf{t}, \xi, \bar{\xi}) = -\mathsf{R} \sum_{\ell,m} \frac{1}{\ell(\ell-1)} h_\ell \mathcal{E}_{\ell,m}(\mathsf{t}) Y_{\ell,m}(\xi, \bar{\xi}), \tag{5.1.10}$$

where the surficial Love numbers h_ℓ are dimensionless and equivalent scale-free measures of the deformation. We can further relate to the 2-dimensional Ricci scalar ${}^{(2)}R$ that describes the intrinsic geometry, and its appropriate perturbation due to the tidal deformation, as

$${}^{(2)}\hat{R}(\mathsf{t}, \xi, \bar{\xi}) = -\frac{2}{\mathsf{R}^2} \sum_{\ell,m} \frac{\ell+2}{\ell} h_\ell \mathcal{E}_{\ell,m}(\mathsf{t}) Y_{\ell,m}(\xi, \bar{\xi}). \tag{5.1.11}$$

We extend the definition to the relativistic limit by the appropriate notion of the multipole field in a coordinate-invariant manner.

5.1.4 Electromagnetic Polarizability

The polarizability of an object is the constant of proportionality between the external field and the induced multipole moment in the body under consideration, similar to the definition of gravitational Love numbers.

5.2 Tidal Perturbations of Isolated Horizons

Modeling a tidally perturbed black hole horizon as an isolated horizon, we construct a perturbed NEH by the geometry of the cross-section, which will be embedded within the horizon.

5.2.1 Intrinsic Geometry

The generic metric within a distorted topological 2-sphere cross-section can be written in holomorphic coordinates $\{\xi, \bar{\xi}\}$ from Eq. (2.4.15), as,

$$ds^2 \doteq \frac{2\mathsf{R}^2}{|P|^2} d\xi d\bar{\xi}. \tag{5.2.1}$$

For a generic axi-symmetric metric, perturbations are translated to perturbations of P even when the perturbation is not axi-symmetric. Starting from $P_{[0]}$, we shall perturb the 2-metric by,

$$P \doteq P_{[0]}\left(1+\hat{P}\right), \tag{5.2.2}$$

where $\hat{P}$ is a small perturbation, which we expand up to linear order.

The directional derivative δ and the connection coefficient a describe the intrinsic geometry through the derivative operator $\hat{\nabla}$. We have the complex null form from Eq. (2.4.14), $\boldsymbol{m} \doteq \frac{\mathsf{R}}{P} d\xi$, such that, the directional derivative,

$$\begin{aligned}\delta &\doteq \frac{1}{\mathsf{R}} P_{[0]}\left(1+\hat{P}\right)\frac{\partial}{\partial \xi} \\ &\doteq \left(1+\hat{P}\right)\delta_{[0]},\end{aligned} \tag{5.2.3}$$

where $\delta_{[0]} \doteq \frac{P_{[0]}}{\mathsf{R}}\frac{\partial}{\partial \xi}$, and the perturbation, $\hat{\delta} \doteq \hat{P}\delta_{[0]}$

The perturbed connection coefficient for the angular derivatives, from Eq. (2.4.17) is $a \doteq \frac{1}{\mathsf{R}}\frac{\partial \bar{P}}{\partial \xi}$, such that,

$$a \doteq \frac{1}{\mathsf{R}}\frac{\partial}{\partial \xi}\left[\bar{P}_{[0]}\left(1+\hat{\bar{P}}\right)\right] \doteq a_{[0]} + a_{[0]}\hat{\bar{P}} + \frac{\bar{P}_{[0]}}{\mathsf{R}}\frac{\partial \hat{\bar{P}}}{\partial \xi} \tag{5.2.4}$$

where $a_{[0]} \doteq \frac{1}{\mathsf{R}}\frac{\partial \bar{P}_{[0]}}{\partial \xi}$.

To determine the intrinsic geometry and curvature, we expand the Ricci scalar, upto linear order, ${}^{(2)}R \doteq {}^{(2)}R_{[0]} + {}^{(2)}\hat{R}$, with the unperturbed curvature ${}^{(2)}R_{[0]} \doteq \Delta_{[0]} \ln\left|P_{[0]}\right|^2$ using Eq. (2.4.18), given by

$$\begin{aligned}{}^{(2)}R &\doteq \Delta_P \ln\left|P_{[0]}\left(1+\hat{P}\right)\right|^2 \doteq \Delta_P \ln\left|P_{[0]}\right|^2 + 2\Delta_P \ln\left(1+\hat{P}\right) \\ &\doteq \Delta_{[0]} \ln\left|P_{[0]}\right|^2 + 2\,\mathrm{Re}\,\hat{P}\Delta_{[0]}\ln\left|P_{[0]}\right|^2 + 2\Delta_P\hat{P} + \mathcal{O}(\hat{P})^2 \\ &\doteq {}^{(2)}R_{[0]} + 2\,\mathrm{Re}\,\hat{P}\,{}^{(2)}R_{[0]} + 2\Delta_{[0]}\,\mathrm{Re}\,\hat{P} + \mathcal{O}(\hat{P})^2,\end{aligned} \tag{5.2.5a}$$

since $\Delta_P \doteq \frac{2|P|^2}{\mathsf{R}^2}\frac{\partial^2}{\partial\xi\partial\bar{\xi}} \doteq \left(1+2\,\mathrm{Re}\,\hat{P}\right)\Delta_{[0]} + \mathcal{O}(\hat{P})^2$, hence

$${}^{(2)}\hat{R} \doteq 2\Delta_{[0]}\,\mathrm{Re}\,\hat{P} + 2\,\mathrm{Re}\,\hat{P}\,{}^{(2)}R_{[0]}. \tag{5.2.5b}$$

Further, using the angular field equations from Eq. (2.2.26), in Eq. (2.4.4), we can derive the perturbations in the intrinsic geometry, governed by $\Psi_2 \doteq \Psi_2^0 + \hat{\Psi}_2$, as,

$$ {}^{(2)}\hat{R} \doteq -4\,\mathrm{Re}\,\hat{\Psi}_2, \tag{5.2.6} $$

since $\rho \doteq \sigma \doteq \gamma \doteq 0$ and $\mu \doteq \bar{\mu}$ on the horizon from Eq. (4.1.5), Eq. (4.1.8) and Eq. (4.1.9), and we assume vacuum spacetime.

Further, we use the angular field equations in Eq. (2.2.26), which yield expressions for the perturbation to the Weyl scalar Ψ_2 as a function of the perturbed connection a, and the spin coefficient π,

$$ -\Psi_2 \doteq \delta a + \bar{\delta}\bar{a} + \delta\bar{\beta} - \bar{\delta}\bar{\alpha} - \alpha\bar{a} + \beta a, \tag{5.2.7a} $$

hence, we have the real and imaginary parts of the Weyl tensor describing the intrinsic geometry,

$$ \begin{aligned} -2\,\mathrm{Re}\,\Psi_2 &\doteq \delta a + \bar{\delta}(\bar{\alpha} - \beta) - 2|a|^2 \\ &\doteq -2\,\mathrm{Re}\,\Psi_2^{[0]} + \delta_{[0]}\hat{a} + \hat{\delta} a_{[0]} + \bar{\delta}_{[0]}\hat{\bar{a}} + \hat{\bar{\delta}}\bar{a}_{[0]} + 2\hat{a}\bar{a}_{[0]} + 2a_{[0]}\hat{\bar{a}}, \qquad (5.2.7\mathrm{b}) \\ -2\,\mathrm{Im}\,\Psi_2 &\doteq \delta\pi - \bar{\delta}\bar{\pi} - \pi(\bar{\alpha} - \beta) + \bar{\pi} a \\ &\doteq -2\,\mathrm{Im}\,\Psi_2^{[0]} + \delta_{[0]}\hat{\pi} + \bar{\delta}\pi_{[0]} - \bar{\delta}_{[0]}\hat{\bar{\pi}} - \hat{\bar{\delta}}\bar{\pi}_{[0]} - \hat{\pi}\bar{a}_{[0]} - \pi_{[0]}\hat{\bar{a}} \\ &\quad + \hat{\bar{\pi}} a_{[0]} + \bar{\pi}_{[0]}\hat{a}, \qquad (5.2.7\mathrm{c}) \end{aligned} $$

where we have used the perturbed relations for the connection and directional derivative from Eq. (5.2.3) and Eq. (5.2.4).

Embedding NEH

To construct a perturbed NEH, we have to provide a foliation of the cross-sections of the horizon in terms of closed 2-spheres. As seen in Sec. 3.3.2, different gauge choices for the rotation 1-form ω correspond to different foliations. For the unperturbed horizon, we have the foliation in Eq. (3.3.3),

$$ \boldsymbol{\omega} \doteq -\star d\mathcal{U} + d\mathcal{V}, \tag{5.2.8a} $$

where we can relate the scalar potentials for the divergence and curl of $\boldsymbol{\omega}$,

$$ \begin{aligned} \Delta_P \mathcal{U} &\doteq -\star d\boldsymbol{\omega} \doteq 2\,\mathrm{Im}\,\Psi_2, & (5.2.8\mathrm{b}) \\ \Delta_P \mathcal{V} &\doteq \hat{\nabla}\cdot\boldsymbol{\omega}. & (5.2.8\mathrm{c}) \end{aligned} $$

,

For a perturbation in the scalar rotational potential, $\mathcal{U} \doteq \mathcal{U}_{[0]} + \hat{\mathcal{U}}$, we expand the derivative through the perturbed conformal factor in Eq. (5.2.2), such that,

$$ \Delta_P \mathcal{U} \doteq \left(1 + 2\,\mathrm{Re}\,\hat{P}\right)\Delta_{[0]}\left(\mathcal{U}_{[0]} + \hat{\mathcal{U}}\right) $$

$$\doteq \Delta_{[0]}\mathcal{U}_{[0]} + 2\,\mathrm{Re}\,\hat{P}\Delta_{[0]}\mathcal{U}_{[0]} + \Delta_{[0]}\hat{\mathcal{U}} + O(\hat{P}^2). \tag{5.2.9a}$$

Thereby, substituting the perturbed Weyl scalar, we have the linear order perturbation,

$$\Delta_{[0]}\hat{\mathcal{U}} \doteq 2\,\mathrm{Im}\,\hat{\Psi}_2 - 4\,\mathrm{Re}\,\hat{P}\,\mathrm{Im}\,\Psi_2^{[0]}, \tag{5.2.9b}$$

In Eq. (5.2.8), $\mathcal{V}$, which is a co-exact form, represents the gauge freedom in the choice of $\boldsymbol{\omega}$. As seen in Sec. 3.3.2, we have a natural choice for the unperturbed horizon with $d\mathcal{V} \doteq 0$. For the perturbed horizon, we choose the gauge choice for the divergence scalar,

$$\Delta_{[0]}\hat{\mathcal{V}} \doteq -2\,\mathrm{Re}\,\hat{\Psi}_2, \tag{5.2.10}$$

which relates the slicing of the perturbed horizon to the intrinsic geometry. This choice guarantees that the vector $\boldsymbol{n}$ and its expansion μ are invariant under the perturbation. The perturbed radial coordinate coincides with the unperturbed one.

Perturbations of Intrinsic Geometry

The evolution of the spin-coefficients and their relations with the Weyl tensors are given by the field equations in Eq. (4.1.16) for the vacuum spacetime. We take the first-order linear perturbations and simplify extensively by substituting the spin coefficients and Weyl tensors. For an isolated horizon, we have the unperturbed spin coefficients, $\tau_{[0]} = \nu_{[0]} = \gamma_{[0]} = 0$ from Eq. (4.1.5) generically on the spacetime, $\mu_{[0]} = \bar{\mu}_{[0]}$ and $\pi_{[0]} = \alpha_{[0]} + \bar{\beta}_{[0]}$ from Eq. (4.1.9), and $\kappa_{[0]} \doteq \rho_{[0]} \doteq \sigma_{[0]} \doteq 0$ using Eq. (4.1.9) and Eq. (4.1.10).

Additionally, we shall choose gauge conditions such that $\hat{\pi} \doteq \hat{\alpha} + \hat{\bar{\beta}}$ on the horizon, up to first order, motivated by the gauge conditions for the intrinsic quantities on the horizon.

Under these constraints, we have the evolution equations from Eq. (4.1.16) on the horizon,

$$D_{[0]}\hat{\rho} - \bar{\delta}_{[0]}\hat{\kappa} \doteq \hat{\rho}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) - 2\alpha_{[0]}\hat{\kappa}, \tag{5.2.11a}$$

$$D_{[0]}\hat{\sigma} - \delta_{[0]}\hat{\kappa} \doteq \hat{\sigma}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) - 2\bar{\beta}_{[0]}\hat{\kappa} + \hat{\Psi}_0, \tag{5.2.11b}$$

$$D_{[0]}\hat{\lambda} - \bar{\delta}_{[0]}\hat{\pi} \doteq -2\epsilon_{[0]}\hat{\lambda} + \mu_{[0]}\hat{\bar{\sigma}} + 2\alpha_{[0]}\hat{\pi}, \tag{5.2.11c}$$

$$D_{[0]}\hat{\mu} - \delta_{[0]}\hat{\pi} \doteq -\hat{\mu}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) + \mu_{[0]}\left(\hat{\bar{\rho}} - \hat{\epsilon} - \hat{\bar{\epsilon}}\right) + \hat{\Psi}_2, \tag{5.2.11d}$$

$$\begin{aligned} D_{[0]}\hat{\alpha} - \bar{\delta}_{[0]}\hat{\epsilon} \doteq{}& \hat{\bar{\delta}}\epsilon_{[0]} + \hat{\alpha}\left(\bar{\epsilon}_{[0]} - 2\epsilon_{[0]}\right) + \alpha_{[0]}\left(\hat{\rho} + \hat{\epsilon} - 2\hat{\bar{\epsilon}}\right) \\ &+ \beta_{[0]}\hat{\bar{\sigma}} - \epsilon_{[0]}\hat{\bar{\beta}} - \bar{\beta}_{[0]}\hat{\epsilon} + \epsilon_{[0]}\hat{\pi}, \end{aligned} \tag{5.2.11e}$$

$$\begin{aligned} D_{[0]}\hat{\beta} - \delta_{[0]}\hat{\epsilon} \doteq{}& \hat{\delta}\epsilon_{[0]} + \alpha_{[0]}\hat{\sigma} - \epsilon_{[0]}\hat{\beta} + \beta_{[0]}\left(\hat{\bar{\rho}} - \hat{\epsilon}\right) - \mu_{[0]}\hat{\kappa} \\ &- \bar{\alpha}_{[0]}\hat{\epsilon} + \epsilon_{[0]}\left(\hat{\bar{\pi}} - \hat{\bar{\alpha}}\right) + \hat{\Psi}_1, \end{aligned} \tag{5.2.11f}$$

where we have assumed the directional derivative D remains unperturbed on the horizon, such that $D \doteq D_{[0]}$.

The angular field equations from Eq. (4.1.16) on the horizon, in linear order perturbations, are given as,

$$\delta_{[0]}\hat{\rho} - \bar{\delta}_{[0]}\sigma \doteq \hat{\sigma}\left(\bar{\beta}_{[0]} - 3\alpha_{[0]}\right) - \hat{\Psi}_1, \tag{5.2.11g}$$

$$\delta_{[0]}\hat{\lambda} - \bar{\delta}_{[0]}\hat{\mu} - \hat{\bar{\delta}}\mu_{[0]} \doteq \mu_{[0]}\hat{\pi} + \hat{\lambda}\left(\bar{\alpha}_{[0]} - 3\beta_{[0]}\right) - \hat{\Psi}_3, \tag{5.2.11h}$$

$$\delta_{[0]}\hat{\alpha} + \hat{\delta}\alpha_{[0]} - \bar{\delta}_{[0]}\hat{\beta} - \hat{\bar{\delta}}\beta_{[0]} \doteq \mu_{[0]}\hat{\rho} + \alpha_{[0]}\left(\hat{\bar{\alpha}} - 3\hat{\beta}\right) - \beta_{[0]}\left(3\hat{\alpha} - \hat{\bar{\beta}}\right) - \hat{\Psi}_2, \tag{5.2.11i}$$

where we used the relations $\hat{\mu} = \hat{\bar{\mu}}$, since μ is real from Eq. (4.1.8).

Further, we can perturb the evolutionary Bianchi identities in Eq. (4.1.17) up to first order, assuming vacuum spacetime, to arrive at the evolution equations for the Weyl tensors,

$$D_{[0]}\hat{\Psi}_1 \doteq \bar{\delta}_{[0]}\hat{\Psi}_0 - 4\alpha_{[0]}\hat{\Psi}_0 + 2\epsilon_{[0]}\hat{\Psi}_1 - 3\hat{\kappa}\Psi_2^{[0]}, \tag{5.2.12a}$$

$$D_{[0]}\hat{\Psi}_2 \doteq \bar{\delta}_{[0]}\hat{\Psi}_1 - 2\alpha_{[0]}\hat{\Psi}_1 + 3\hat{\rho}\Psi_2^{[0]}, \tag{5.2.12b}$$

$$D_{[0]}\hat{\Psi}_3 \doteq \bar{\delta}_{[0]}\hat{\Psi}_2 + \hat{\bar{\delta}}_{[0]}\Psi_2^{[0]} + 3\hat{\pi}\Psi_2^{[0]} - 2\epsilon_{[0]}\hat{\Psi}_3, \tag{5.2.12c}$$

$$D_{[0]}\hat{\Psi}_4 \doteq \bar{\delta}_{[0]}\hat{\Psi}_3 - 3\Psi_2^{[0]}\hat{\lambda} + 2\alpha_{[0]}\hat{\Psi}_3 - 4\epsilon_{[0]}\hat{\Psi}_4. \tag{5.2.12d}$$

5.2.2 Perturbative Radial Evolution

We propagate the tetrad basis and the coordinate system defined at the horizon by parallel propagating all fields along the inward-pointing future-directed null vector $\boldsymbol{n}$, such that the radial derivative remains unchanged $\Delta = \Delta_{[0]}$.

The radial field equations from Eq. (4.1.16) in the neighborhood of the horizon, for the spin coefficients are simplified in linear perturbations as,

$$\Delta\hat{\pi} = -\pi_{[0]}\hat{\mu} - \mu_{[0]}\hat{\pi} - \lambda_{[0]}\hat{\bar{\pi}} - \pi_{[0]}\hat{\lambda} - \hat{\Psi}_3, \tag{5.2.13a}$$

$$\Delta\hat{\beta} = -\mu_{[0]}\hat{\beta} - \beta_{[0]}\hat{\mu} - \bar{\lambda}_{[0]}\hat{\alpha} - \alpha_{[0]}\hat{\bar{\lambda}} \tag{5.2.13b}$$

$$\Delta\hat{\alpha} = -\lambda_{[0]}\hat{\beta} - \beta_{[0]}\hat{\lambda} - \alpha_{[0]}\hat{\mu} - \mu_{[0]}\hat{\alpha} - \hat{\Psi}_3, \tag{5.2.13c}$$

$$\Delta\hat{\lambda} = -2\lambda_{[0]}\hat{\mu} - 2\mu_{[0]}\hat{\lambda} - \hat{\Psi}_4, \tag{5.2.13d}$$

$$\Delta\hat{\mu} = -2\mu_{[0]}\hat{\mu} - \lambda_{[0]}\hat{\bar{\lambda}} - \bar{\lambda}_{[0]}\hat{\lambda}, \tag{5.2.13e}$$

$$\Delta\hat{\rho} = -\mu_{[0]}\hat{\rho} - \rho_{[0]}\hat{\mu} - \sigma_{[0]}\hat{\lambda} - \lambda_{[0]}\hat{\sigma} - \hat{\Psi}_2, \tag{5.2.13f}$$

$$\Delta\hat{\sigma} = -\mu_{[0]}\hat{\sigma} - \sigma_{[0]}\hat{\mu} - \bar{\lambda}_{[0]}\hat{\rho} - \rho_{[0]}\hat{\bar{\lambda}}, \tag{5.2.13g}$$

$$\Delta\hat{\kappa} = -\rho_{[0]}\hat{\bar{\pi}} - \bar{\pi}_{[0]}\hat{\rho} - \sigma_{[0]}\hat{\pi} - \pi_{[0]}\hat{\sigma} - \hat{\Psi}_1, \tag{5.2.13h}$$

$$\Delta\hat{\epsilon} = -\alpha_{[0]}\hat{\bar{\pi}} - \bar{\pi}_{[0]}\hat{\alpha} - \beta_{[0]}\hat{\pi} - \pi_{[0]}\hat{\beta} - \hat{\Psi}_2, \tag{5.2.13i}$$

where we used the relations $\hat{\tau} = \hat{\nu} = \hat{\gamma} = 0$, motivated from Eq. (4.1.5).

The radial evolution of the Weyl tensors is perturbatively specified by the radial Bianchi identities in Eq. (4.1.17), which we expand in linear order as,

$$\Delta\hat{\Psi}_0 = \delta_{[0]}\hat{\Psi}_1 - \mu_{[0]}\hat{\Psi}_0 - \Psi_0^{[0]}\hat{\mu} - 2\beta_{[0]}\hat{\Psi}_1 - 2\Psi_1^{[0]}\hat{\beta} + 3\sigma_{[0]}\hat{\Psi}_2 + 3\Psi_2^{[0]}\hat{\sigma}, \tag{5.2.14a}$$

$$\Delta\hat{\Psi}_1 = \delta_{[0]}\hat{\Psi}_2 - 2\mu_{[0]}\hat{\Psi}_1 - 2\Psi_1^{[0]}\hat{\mu} + 2\sigma_{[0]}\hat{\Psi}_3 + 2\Psi_3^{[0]}\hat{\sigma}, \tag{5.2.14b}$$

$$\Delta\hat{\Psi}_2 = \delta_{[0]}\hat{\Psi}_3 - 3\mu_{[0]}\hat{\Psi}_2 - 3\Psi_2^{[0]}\hat{\mu} + 2\beta_{[0]}\hat{\Psi}_3 + 2\Psi_3^{[0]}\hat{\beta} + \sigma_{[0]}\hat{\Psi}_4 + \Psi_4^{[0]}\hat{\sigma}, \tag{5.2.14c}$$

$$\Delta\hat{\Psi}_3 = \delta_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\hat{\Psi}_3 - 4\Psi_3^{[0]}\hat{\mu} + 4\beta_{[0]}\hat{\Psi}_4 + 4\Psi_4^{[0]}\hat{\beta}. \tag{5.2.14d}$$

Thereby, we can evaluate the perturbations of the spin coefficients and the Weyl tensors to reconstruct the perturbed metric through the radial, temporal, and angular evolution of the frame functions.

5.2.3 Electromagnetic Fields

We proceed with considering the electromagnetic fields. We use the unperturbed spin coefficients for an isolated horizon as earlier, with $\tau_{[0]} = \nu_{[0]} = \gamma_{[0]} = 0$ from Eq. (4.1.5), $\mu_{[0]} = \bar{\mu}_{[0]}$ and $\pi_{[0]} = \alpha_{[0]} + \bar{\beta}_{[0]}$ from Eq. (4.1.9), and $\kappa_{[0]} \doteq \rho_{[0]} \doteq \sigma_{[0]} \doteq 0$ using Eq. (4.1.9) and Eq. (4.1.10).

Intrinsic Geometry

In terms of the directional derivatives, the evolutionary Maxwell equations in Eq. (2.2.32) can be expanded on the horizon with $D \doteq D_{[0]}$ and assuming source-free spacetime,

$$\begin{aligned} D_{[0]}\hat{\phi}_1 - \bar{\delta}_{[0]}\hat{\phi}_0 - \hat{\bar{\delta}}\phi_0^{[0]} &\doteq \phi_0^{[0]}(\hat{\pi} - 2\hat{\alpha}) + \hat{\phi}_0\left(\pi_{[0]} - 2\alpha_{[0]}\right) + 2\rho_{[0]}\hat{\phi}_1 \\ &\quad + 2\phi_1^{[0]}\hat{\rho} - \kappa_{[0]}\hat{\phi}_2 - \phi_2^{[0]}\hat{\kappa}, \end{aligned} \tag{5.2.15a}$$

$$\begin{aligned} D_{[0]}\hat{\phi}_2 - \bar{\delta}_{[0]}\hat{\phi}_1 - \hat{\bar{\delta}}\phi_1^{[0]} &= \phi_2^{[0]}(\hat{\rho} - 2\hat{\epsilon}) + \hat{\phi}_2\left(\rho_{[0]} - 2\epsilon_{[0]}\right) - \lambda_{[0]}\hat{\phi}_0 - \phi_0^{[0]}\hat{\lambda} \\ &\quad + 2\pi_{[0]}\hat{\phi}_1 + 2\phi_1^{[0]}\hat{\pi}. \end{aligned} \tag{5.2.15b}$$

The perturbations of the evolution equations in Eq. (2.2.26) containing the field tensors are simplified on the horizon, analogous to Eq. (5.2.11)

$$D_{[0]}\hat{\rho} - \bar{\delta}_{[0]}\hat{\kappa} \doteq \hat{\rho}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) - 2\alpha_{[0]}\hat{\kappa} + 2\bar{\phi}_0^{[0]}\hat{\phi}_0 + 2\phi_0^{[0]}\hat{\bar{\phi}}_0, \tag{5.2.16a}$$

$$D_{[0]}\hat{\sigma} - \delta_{[0]}\hat{\kappa} \doteq \hat{\sigma}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) - 2\bar{\beta}_{[0]}\hat{\kappa} + \hat{\Psi}_0, \tag{5.2.16b}$$

$$D_{[0]}\hat{\lambda} - \bar{\delta}_{[0]}\hat{\pi} \doteq -2\epsilon_{[0]}\hat{\lambda} + \mu_{[0]}\hat{\bar{\sigma}} + 2\alpha_{[0]}\hat{\pi} + 2\phi_2^{[0]}\hat{\bar{\phi}}_0 + 2\bar{\phi}_0^{[0]}\hat{\phi}_2, \tag{5.2.16c}$$

$$D_{[0]}\hat{\mu} - \delta_{[0]}\hat{\pi} \doteq -\hat{\mu}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) + \mu_{[0]}\left(\hat{\rho} - \hat{\epsilon} - \hat{\bar{\epsilon}}\right) + \hat{\Psi}_2, \tag{5.2.16d}$$

$$\begin{aligned} D_{[0]}\hat{\alpha} - \bar{\delta}_{[0]}\hat{\epsilon} \doteq{}& \hat{\bar{\delta}}\epsilon_{[0]} + \hat{\alpha}\left(\bar{\epsilon}_{[0]} - 2\epsilon_{[0]}\right) + \alpha_{[0]}\left(\hat{\rho} + \hat{\epsilon} - 2\hat{\bar{\epsilon}}\right) \\ &+ \beta_{[0]}\hat{\bar{\sigma}} - \epsilon_{[0]}\hat{\bar{\beta}} - \bar{\beta}_{[0]}\hat{\epsilon} + \epsilon_{[0]}\hat{\pi} + 2\phi_1^{[0]}\hat{\bar{\phi}}_0 + 2\bar{\phi}_0^{[0]}\hat{\phi}_1, \end{aligned} \tag{5.2.16e}$$

$$\begin{aligned} D_{[0]}\hat{\beta} - \delta_{[0]}\hat{\epsilon} \doteq{}& \hat{\delta}\epsilon_{[0]} + \alpha_{[0]}\hat{\sigma} - \epsilon_{[0]}\hat{\beta} + \beta_{[0]}\left(\hat{\bar{\rho}} - \hat{\epsilon}\right) - \mu_{[0]}\hat{\kappa} \\ &- \bar{\alpha}_{[0]}\hat{\epsilon} + \epsilon_{[0]}\left(\hat{\bar{\pi}} - \hat{\bar{\alpha}}\right) + \hat{\Psi}_1, \end{aligned} \tag{5.2.16f}$$

The angular field equations incorporate the perturbed Maxwell's fields from Eq. (2.2.34) as,

$$\delta_{[0]}\hat{\rho} - \bar{\delta}_{[0]}\sigma \doteq \hat{\sigma}\left(\bar{\beta}_{[0]} - 3\alpha_{[0]}\right) - \hat{\Psi}_1 + 2\phi_0^{[0]}\hat{\bar{\phi}}_1 + 2\bar{\phi}_1^{[0]}\hat{\phi}_0, \tag{5.2.16g}$$

$$\delta_{[0]}\hat{\lambda} - \bar{\delta}_{[0]}\hat{\mu} - \hat{\bar{\delta}}\mu_{[0]} \doteq \mu_{[0]}\hat{\pi} + \hat{\lambda}\left(\bar{\alpha}_{[0]} - 3\beta_{[0]}\right) - \hat{\Psi}_3 + 2\phi_2^{[0]}\hat{\bar{\phi}}_1 + 2\bar{\phi}_1^{[0]}\hat{\phi}_2, \tag{5.2.16h}$$

$$\begin{aligned} \delta_{[0]}\hat{\alpha} + \hat{\delta}\alpha_{[0]} - \bar{\delta}_{[0]}\hat{\beta} - \hat{\bar{\delta}}\beta_{[0]} \doteq{}& \mu_{[0]}\hat{\rho} + \alpha_{[0]}\left(\hat{\bar{\alpha}} - 3\hat{\beta}\right) - \beta_{[0]}\left(3\hat{\alpha} - \hat{\bar{\beta}}\right) \\ &- \hat{\Psi}_2 + 2\bar{\phi}_1^{[0]}\hat{\phi}_1 + 2\phi_1^{[0]}\hat{\bar{\phi}}_1. \end{aligned} \tag{5.2.16i}$$

Further, using Eq. (2.2.35), the evolutionary Bianchi identities in Eq. (5.2.12) become,

$$\begin{aligned} D_{[0]}\hat{\Psi}_1 \doteq{}& \bar{\delta}_{[0]}\hat{\Psi}_0 - 4\alpha_{[0]}\hat{\Psi}_0 + 2\epsilon_{[0]}\hat{\Psi}_1 - 3\hat{\kappa}\Psi_2^{[0]} \\ &- 2\bar{\phi}_0^{[0]}\delta_{[0]}\hat{\phi}_0 - 2\bar{\phi}_0^{[0]}\hat{\delta}\phi_0^{[0]} - 2\bar{\delta}_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_0 \\ &+ 2\bar{\phi}_1^{[0]}D_{[0]}\hat{\phi}_0 + 2D_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_1 - 4\epsilon_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 \\ &- 4\epsilon_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 - 4\left|\phi_1^{[0]}\right|^2\hat{\epsilon} - 4\sigma_{[0]}\bar{\phi}_0^{[0]}\hat{\phi}_1 \\ &- 4\sigma_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_0 - 4\bar{\phi}_0^{[0]}\phi_1^{[0]}\hat{\sigma} + 4\kappa_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 \\ &+ 4\kappa_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 + 4\left|\phi_1^{[0]}\right|^2\hat{\kappa} + 4\beta_{[0]}\bar{\phi}_0^{[0]}\hat{\phi}_0 \\ &+ 4\beta_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_0 + 4\left|\phi_0^{[0]}\right|^2\hat{\beta}_{[0]}, \end{aligned} \tag{5.2.17a}$$

$$\begin{aligned} D_{[0]}\hat{\Psi}_2 \doteq{}& \bar{\delta}_{[0]}\hat{\Psi}_1 - 2\alpha_{[0]}\hat{\Psi}_1 + 3\hat{\rho}\Psi_2^{[0]} \\ &+ 2\bar{\phi}_1^{[0]}\bar{\delta}_{[0]}\hat{\phi}_0 + 2\bar{\phi}_1^{[0]}\hat{\bar{\delta}}\phi_0^{[0]} + 2\bar{\delta}_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_1 \\ &- 2\bar{\phi}_0^{[0]}\Delta_{[0]}\hat{\phi}_0 - 2\Delta_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_0 - 4\alpha_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_0 \\ &- 4\alpha_{[0]}\hat{\bar{\phi}}_1\phi_0^{[0]} - 4\bar{\phi}_1^{[0]}\phi_0^{[0]}\hat{\alpha} + 4\rho_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 \end{aligned}$$

$$
\begin{aligned}
&+ 4\rho_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 + 4\left|\phi_1^{[0]}\right|^2 \hat{\rho},
\end{aligned}
\tag{5.2.17b}
$$

$$
\begin{aligned}
D_{[0]}\hat{\Psi}_3 \doteq{}& \bar{\delta}_{[0]}\hat{\Psi}_2 + \hat{\bar{\delta}}_{[0]}\Psi_2^{[0]} + 3\hat{\pi}\Psi_2^{[0]} - 2\epsilon_{[0]}\hat{\Psi}_3 \\
&- 2\bar{\phi}_0^{[0]}\delta_{[0]}\hat{\phi}_2 - 2\bar{\phi}_0^{[0]}\hat{\delta}\phi_2^{[0]} - 2\delta_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_0 \\
&+ 2\bar{\phi}_1^{[0]}D_{[0]}\hat{\phi}_2 + 2D_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_1 + 4\epsilon_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_2 \\
&+ 4\epsilon_{[0]}\hat{\bar{\phi}}_1\phi_2^{[0]} + 4\bar{\phi}_1^{[0]}\phi_2^{[0]}\hat{\epsilon} + 4\mu_{[0]}\bar{\phi}_0^{[0]}\hat{\phi}_1 \\
&+ 4\mu_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_0 + 4\bar{\phi}_0^{[0]}\phi_1^{[0]}\hat{\mu} - 4\beta_{[0]}\bar{\phi}_0^{[0]}\hat{\phi}_2 \\
&- 4\beta_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_0 - 4\bar{\phi}_0^{[0]}\phi_2^{[0]}\hat{\beta} - 4\pi_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 \\
&- 4\pi_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 - 4\left|\phi_1^{[0]}\right|^2 \hat{\pi},
\end{aligned}
\tag{5.2.17c}
$$

$$
\begin{aligned}
D_{[0]}\hat{\Psi}_4 \doteq{}& \bar{\delta}_{[0]}\hat{\Psi}_3 - 3\Psi_2^{[0]}\hat{\lambda} + 2\alpha_{[0]}\hat{\Psi}_3 - 4\epsilon_{[0]}\hat{\Psi}_4 \\
&+ 2\bar{\phi}_1^{[0]}\bar{\delta}_{[0]}\hat{\phi}_2 + 2\bar{\phi}_1^{[0]}\hat{\bar{\delta}}\phi_2^{[0]} + 2\bar{\delta}_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_1 \\
&- 2\bar{\phi}_0^{[0]}\Delta_{[0]}\hat{\phi}_2 - 2\Delta_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_0 + 4\alpha_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_2 \\
&+ 4\alpha_{[0]}\hat{\bar{\phi}}_1\phi_2^{[0]} + 4\bar{\phi}_1^{[0]}\phi_2^{[0]}\hat{\alpha} - 4\lambda_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 \\
&- 4\lambda_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 - 4\left|\phi_1^{[0]}\right|^2 \hat{\lambda}.
\end{aligned}
\tag{5.2.17d}
$$

Radial Evolution

The radial Maxwell equations in Eq. (2.2.32) for the electromagnetic field, in first-order perturbations, are,

$$\Delta\phi_1 = \delta_{[0]}\hat{\phi}_2 + \hat{\delta}_{[0]}\phi_2^{[0]} + 2\phi_2^{[0]}\hat{\beta} + 2\beta_{[0]}\hat{\phi}_2 + 2\mu_{[0]}\hat{\phi}_1 + 2\phi_1^{[0]}\hat{\mu}, \tag{5.2.18a}$$

$$\Delta\phi_0 = \delta_{[0]}\hat{\phi}_1 + \hat{\delta}_{[0]}\phi_1^{[0]} - \mu_{[0]}\hat{\phi}_0 - \phi_0^{[0]}\hat{\mu} + \sigma_{[0]}\hat{\phi}_2 + \phi_2^{[0]}\hat{\sigma}, \tag{5.2.18b}$$

Further, to evaluate the spin coefficients, the radial equations in Eq. (5.2.13) include the perturbations of Maxwell's fields from Eq. (2.2.34) as,

$$\Delta\hat{\pi} = -\pi_{[0]}\hat{\mu} - \mu_{[0]}\hat{\pi} - \lambda_{[0]}\hat{\bar{\pi}} - \pi_{[0]}\hat{\lambda} - \hat{\Psi}_3 - 2\phi_2^{[0]}\hat{\bar{\phi}}_1 - 2\bar{\phi}_1^{[0]}\hat{\phi}_2, \tag{5.2.19a}$$

$$\Delta\hat{\beta} = -\mu_{[0]}\hat{\beta} - \beta_{[0]}\hat{\mu} - \bar{\lambda}_{[0]}\hat{\alpha} - \alpha_{[0]}\hat{\bar{\lambda}} - 2\bar{\phi}_2^{[0]}\hat{\phi}_1 - 2\phi_1^{[0]}\hat{\bar{\phi}}_2, \tag{5.2.19b}$$

$$\Delta\hat{\alpha} = -\lambda_{[0]}\hat{\beta} - \beta_{[0]}\hat{\lambda} - \alpha_{[0]}\hat{\mu} - \mu_{[0]}\hat{\alpha} - \hat{\Psi}_3, \tag{5.2.19c}$$

$$\Delta\hat{\lambda} = -2\lambda_{[0]}\hat{\mu} - 2\mu_{[0]}\hat{\lambda} - \hat{\Psi}_4, \tag{5.2.19d}$$

$$\Delta\hat{\mu} = -2\mu_{[0]}\hat{\mu} - \lambda_{[0]}\hat{\bar{\lambda}} - \bar{\lambda}_{[0]}\hat{\lambda} - 2\bar{\phi}_2^{[0]}\hat{\phi}_2 - 2\phi_2^{[0]}\hat{\bar{\phi}}_2, \tag{5.2.19e}$$

$$\Delta\hat{\rho} = -\mu_{[0]}\hat{\rho} - \rho_{[0]}\hat{\mu} - \sigma_{[0]}\hat{\lambda} - \lambda_{[0]}\hat{\sigma} - \hat{\Psi}_2, \tag{5.2.19f}$$

$$\Delta\hat{\sigma} = -\mu_{[0]}\hat{\sigma} - \sigma_{[0]}\hat{\mu} - \bar{\lambda}_{[0]}\hat{\rho} - \rho_{[0]}\hat{\bar{\lambda}} - 2\bar{\phi}_2^{[0]}\hat{\phi}_0 - 2\phi_0^{[0]}\hat{\bar{\phi}}_2, \tag{5.2.19g}$$

$$\Delta\hat{\kappa} = -\rho_{[0]}\hat{\bar{\pi}} - \bar{\pi}_{[0]}\hat{\rho} - \sigma_{[0]}\hat{\pi} - \pi_{[0]}\hat{\sigma} - \hat{\Psi}_1 - 2\bar{\phi}_1^{[0]}\hat{\phi}_0 - 2\phi_0^{[0]}\hat{\bar{\phi}}_1, \tag{5.2.19h}$$

$$\Delta\hat{\epsilon} = -\alpha_{[0]}\hat{\bar{\pi}} - \bar{\pi}_{[0]}\hat{\alpha} - \beta_{[0]}\hat{\pi} - \pi_{[0]}\hat{\beta} - \hat{\Psi}_2 - 2\bar{\phi}_1^{[0]}\hat{\phi}_1 - 2\phi_1^{[0]}\hat{\bar{\phi}}_1. \tag{5.2.19i}$$

Further, the radial evolution of the perturbations of the Bianchi identities in Eq. (5.2.14) is modified using the electromagnetic field perturbations through Eq. (2.2.35) as,

$$\begin{aligned}
\Delta\hat{\Psi}_0 = {}& \delta_{[0]}\hat{\Psi}_1 - \mu_{[0]}\hat{\Psi}_0 - \Psi_0^{[0]}\hat{\mu} - 2\beta_{[0]}\hat{\Psi}_1 - 2\Psi_1^{[0]}\hat{\beta} + 3\sigma_{[0]}\hat{\Psi}_2 \\
& + 3\Psi_2^{[0]}\hat{\sigma} + 2\bar{\phi}_1^{[0]}\delta_{[0]}\hat{\phi}_0 + 2\bar{\phi}_1^{[0]}\hat{\delta}\phi_0^{[0]} + 2\delta_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_1 \\
& + 2\bar{\phi}_2^{[0]}D_{[0]}\hat{\phi}_0 + 2\bar{\phi}_2^{[0]}\hat{D}\phi_0^{[0]} + 2D_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_2 - 4\kappa_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_1 \\
& - 4\kappa_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_2 - 4\bar{\phi}_2^{[0]}\phi_1^{[0]}\hat{\kappa} - 4\beta_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_0 - 4\beta_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_1 \\
& - 4\phi_0^{[0]}\bar{\phi}_1^{[0]}\hat{\beta} + 4\sigma_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 - 4\sigma_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 - 4\left|\phi_1^{[0]}\right|^2\hat{\sigma} \\
& + 4\epsilon_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_0 + 4\epsilon_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_2 + 4\bar{\phi}_2^{[0]}\phi_0^{[0]}\hat{\epsilon},
\end{aligned} \tag{5.2.20a}$$

$$\begin{aligned}
\Delta\hat{\Psi}_1 = {}& \delta_{[0]}\hat{\Psi}_2 - 2\mu_{[0]}\hat{\Psi}_1 - 2\Psi_1^{[0]}\hat{\mu} + 2\sigma_{[0]}\hat{\Psi}_3 + 2\Psi_3^{[0]}\hat{\sigma} + 2\bar{\phi}_2^{[0]}\bar{\delta}_{[0]}\hat{\phi}_0 \\
& + 2\bar{\phi}_2^{[0]}\hat{\bar{\delta}}\phi_0^{[0]} + 2\bar{\delta}_{[0]}\phi_2^{[0]}\bar{\phi}_1 + 2\bar{\phi}_2^{[0]}\Delta_{[0]}\hat{\phi}_0 + 2\Delta_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_2 \\
& - 4\rho_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_1 - 4\rho_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_2 - 4\bar{\phi}_2^{[0]}\phi_1^{[0]}\hat{\rho} + 4\alpha_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_0 \\
& + 4\alpha_{[0]}\phi_0^{[0]}\hat{\bar{\phi}}_2 + 4\bar{\phi}_2^{[0]}\phi_0^{[0]}\hat{\alpha},
\end{aligned} \tag{5.2.20b}$$

$$\begin{aligned}
\Delta\hat{\Psi}_2 = {}& \delta_{[0]}\hat{\Psi}_3 - 3\mu_{[0]}\hat{\Psi}_2 - 3\Psi_2^{[0]}\hat{\mu} + 2\beta_{[0]}\hat{\Psi}_3 + 2\Psi_3^{[0]}\hat{\beta} + \sigma_{[0]}\hat{\Psi}_4 \\
& + \Psi_4^{[0]}\hat{\sigma} - 2\bar{\phi}_2^{[0]}D_{[0]}\bar{\phi}_2 - 2\bar{\phi}_2^{[0]}\hat{D}\bar{\phi}_2^{[0]} - 2D_{[0]}\phi_2^{[0]}\bar{\phi}_2^{[0]} + 2\bar{\phi}_1^{[0]}\delta_{[0]}\hat{\phi}_2 \\
& + 2\bar{\phi}_1^{[0]}\hat{\delta}\phi_2^{[0]} + 2\delta_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_1 - 4\epsilon_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_2 - 4\epsilon_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_2 - 4\left|\phi_2^{[0]}\right|^2\hat{\epsilon} \\
& - 4\mu_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_1 - 4\mu_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_1 - 4\left|\phi_1^{[0]}\right|^2\hat{\mu} + 4\beta_{[0]}\bar{\phi}_1^{[0]}\hat{\phi}_2 + 4\beta_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_1 \\
& + 4\bar{\phi}_1^{[0]}\phi_2^{[0]}\hat{\beta} + 4\pi_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_1 + 4\pi_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_2 + 4\bar{\phi}_2^{[0]}\phi_1^{[0]}\hat{\pi},
\end{aligned} \tag{5.2.20c}$$

$$\begin{aligned}
\Delta\hat{\Psi}_3 = {}& \delta_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\hat{\Psi}_3 - 4\Psi_3^{[0]}\hat{\mu} + 4\beta_{[0]}\hat{\Psi}_4 + 4\Psi_4^{[0]}\hat{\beta} \\
& - 2\bar{\phi}_2^{[0]}\bar{\delta}_{[0]}\bar{\phi}_2 - 2\bar{\phi}_2^{[0]}\hat{\bar{\delta}}\bar{\phi}_2^{[0]} - 2\bar{\delta}_{[0]}\phi_2^{[0]}\bar{\phi}_2^{[0]} + 2\bar{\phi}_1^{[0]}\Delta_{[0]}\hat{\phi}_2 \\
& + 2\Delta_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_1 - 4\alpha_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_2 - 4\alpha_{[0]}\phi_2^{[0]}\hat{\bar{\phi}}_2 \\
& - 4\left|\phi_2^{[0]}\right|^2\hat{\alpha} + 4\lambda_{[0]}\bar{\phi}_2^{[0]}\hat{\phi}_1 + 4\lambda_{[0]}\phi_1^{[0]}\hat{\bar{\phi}}_2 + 4\bar{\phi}_2^{[0]}\phi_1^{[0]}\hat{\lambda}.
\end{aligned} \tag{5.2.20d}$$

The remaining contracted Bianchi identities in Eq. (2.2.35) are equivalent to Maxwell's equations since we have the vanishing covariant divergence of the energy-momentum tensor, and hence provide the same physical content.

5.3 Spherical Background

To exploit the explicit symmetries of a spherical static spacetime, we consider the perturbations of a Schwarzschild background, characterized by a spherical cross-section with $P_{[0]} \doteq \frac{1}{\sqrt{2}}\left(1+\xi\bar{\xi}\right)$, as in Eq. (2.4.20). Let us now perturb the conformal factor by $P = P_{[0]}\left(1+\hat{P}\right)$ as per Eq. (5.2.2), where we assume $\hat{P}$ is real for the spherical cross section.

We have the unperturbed solutions for the radial evolution of the spin-coefficients for the background Schwarzschild spacetime from Eq. (4.2.5) and Eq. (4.2.9),

$$\tau_{[0]} = \nu_{[0]} = \lambda_{[0]} = \sigma_{[0]} = \pi_{[0]} = \kappa_{[0]} = 0, \tag{5.3.1a}$$

$$\mu_{[0]} = -\frac{1}{\mathsf{r}}, \tag{5.3.1b}$$

$$\epsilon_{[0]} = \frac{\mathsf{R}}{4\mathsf{r}^2}, \quad \rho_{[0]} = -\frac{1}{2\mathsf{r}}\left(1-\frac{\mathsf{R}}{\mathsf{r}}\right), \tag{5.3.1c}$$

$$a_{[0]} = \frac{a^{(0)}_{[0]}}{\mathsf{r}} = \frac{\bar{\xi}}{\sqrt{2}\mathsf{R}\mathsf{r}} \tag{5.3.1d}$$

where we have used the holomorphic coordinates for the connection.

The appropriate frame functions and the directional derivatives from Eq. (4.2.11) are,

$$D_{[0]} = \partial_{\mathsf{v}} - \left(1-\frac{\mathsf{R}}{\mathsf{r}}\right)\partial_{\mathsf{r}}, \tag{5.3.1e}$$

$$\Delta_{[0]} = -\partial_{\mathsf{r}}, \quad \delta_{[0]} = \frac{P_{[0]}}{\mathsf{r}}\partial_{\xi} \tag{5.3.1f}$$

Further, the unperturbed Weyl tensors at the horizons evolve radially from Eq. (4.2.7), for the Petrov Type D Spacetime as,

$$\Psi_0^{[0]} = \Psi_1^{[0]} = \Psi_3^{[0]} = \Psi_4^{[0]} = 0, \tag{5.3.1g}$$

$$\Psi_2^{[0]} = -\frac{\mathsf{R}}{2\mathsf{r}^3}. \tag{5.3.1h}$$

5.3.1 Intrinsic Perturbations

On the horizon, we can compute the unperturbed directional derivative $\delta_{[0]}$, and the connection coefficient a, using holonomic coordinates, such that $\delta_{[0]} \doteq \frac{P_{[0]}}{\mathsf{R}}\frac{\partial}{\partial\xi} \doteq \frac{1+\xi\bar{\xi}}{\sqrt{2}\mathsf{R}}\frac{\partial}{\partial\xi}$, and $a_{[0]} \doteq \frac{1}{\mathsf{R}}\frac{\partial P_{[0]}}{\partial\xi} \doteq \frac{\bar{\xi}}{\sqrt{2}\mathsf{R}}$ from Eq. (2.4.17). Thereby, the perturbation in Eq. (5.2.4) manifests as,

$$\hat{a} \doteq \frac{\bar{\xi}}{\sqrt{2}\mathsf{R}}\hat{P} + \delta_0\hat{P}. \tag{5.3.2}$$

Further, for the spherical cross-section, since the intrinsic curvature simplifies as ${}^{(2)}R_{[0]} \doteq \Delta_{[0]} \ln \left|P_{[0]}\right|^2 \doteq \frac{2}{\mathsf{R}^2}$ using Eq. (2.4.18), we have the perturbation from Eq. (5.2.5),

$$ {}^{(2)}\hat{R} \doteq 2\Delta_{[0]}\hat{P} + \frac{4}{\mathsf{R}^2}\hat{P}. \tag{5.3.3} $$

Since the metric perturbation is sourced by the perturbation to the real part of Ψ_2 from Eq. (5.2.6), we have,

$$ \Delta_{[0]}\hat{P} + \frac{2}{\mathsf{R}^2}\hat{P} + 2\,\mathrm{Re}\,\hat{\Psi}_2 \doteq 0. \tag{5.3.4} $$

Also, we relate the Weyl scalar to the perturbations in the connection a, and the spin coefficient π, from Eq. (5.2.7), to have,

$$ \begin{aligned} -2\,\mathrm{Re}\,\hat{\Psi}_2 &\doteq \delta_{[0]}\hat{a} + \bar{\delta}_{[0]}\hat{\bar{a}} + \frac{\xi}{\sqrt{2}\mathsf{R}}\hat{a} + \frac{\bar{\xi}}{\sqrt{2}\mathsf{R}}\hat{\bar{a}} \\ &\doteq 2\,\mathrm{Re}\left\{\delta_{[0]}\hat{a} + \frac{\xi}{\sqrt{2}\mathsf{R}}\hat{a}\right\} \doteq 2\,\mathrm{Re}\,\bar{\eth}_{[0]}\hat{a}, \end{aligned} \tag{5.3.5a} $$

$$ \begin{aligned} -2\,\mathrm{Im}\,\hat{\Psi}_2 &\doteq \delta_{[0]}\hat{\pi} - \bar{\delta}_{[0]}\hat{\bar{\pi}} - \frac{\bar{\xi}}{\sqrt{2}\mathsf{R}}\hat{\pi} + \frac{\xi}{\sqrt{2}\mathsf{R}}\hat{\bar{\pi}} \\ &\doteq 2\,\mathrm{Im}\left\{\delta_{[0]}\hat{\pi} + \frac{\xi}{\sqrt{2}\mathsf{R}}\hat{\pi}\right\} \doteq 2\,\mathrm{Im}\,\eth_{[0]}\hat{\pi}, \end{aligned} \tag{5.3.5b} $$

since $\hat{\delta}a_{[0]} \doteq \hat{P}\delta_{[0]}a_{[0]} \doteq 0$ from Eq. (5.2.3).

We have also used the spin operator by the relation in Eq. (2.3.18), defined by $\eth_{[0]} \doteq \delta_{[0]} - \mathfrak{s}\bar{a}_{[0]} \doteq \delta_{[0]} + \frac{\xi}{\sqrt{2}\mathsf{R}}$, for spin-weight $\mathfrak{s} = -1$, which we assume is invariant on perturbation.

Further, since $\mathrm{Im}\,\Psi_2^{[0]} \doteq 0$, we can also relate the perturbed potentials, from Eq. (5.2.9) and Eq. (5.2.10), as

$$ \Delta_{[0]}\hat{\mathcal{U}} \doteq 2\,\mathrm{Im}\,\hat{\Psi}_2 \doteq -2\,\mathrm{Im}\,\eth_{[0]}\hat{\pi}, \tag{5.3.6a} $$

$$ \Delta_{[0]}\hat{\mathcal{V}} \doteq -2\,\mathrm{Re}\,\hat{\Psi}_2 \doteq 2\,\mathrm{Re}\,\bar{\eth}_{[0]}\hat{a}. \tag{5.3.6b} $$

Due to the nature of the intrinsic horizon, we shall work with the gauge choice of $\hat{\epsilon} \doteq \hat{\bar{\epsilon}}$. We also require that the area of the perturbed horizon coincide with the area of the unperturbed horizon, up to linear order, such that we assume the perturbation of the surface gravity at the horizon, $\hat{\kappa}_{(\boldsymbol{\ell})} \doteq 2\hat{\epsilon} \doteq 0$. We also choose the radial null normal to be unperturbed, and restrict $\hat{\mu} \doteq 0$ on the horizon. Further, we require that the flux across the horizon remains zero such that it is isolated, thereby $\hat{\Psi}_0 \doteq \hat{\Psi}_1 \doteq 0$. We further assert that the null normal $\boldsymbol{\ell}$ remains a geodesic on the horizon since it has to be tangential, thus requiring $\hat{\kappa} \doteq 0$.

Thereby, for the spin coefficients, we have the temporal evolution equations from Eq. (5.2.11), in linear order,

$$D_{[0]}\hat{\rho} \doteq \hat{\rho}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) \doteq \frac{1}{2\mathsf{R}}\hat{\rho}, \tag{5.3.7a}$$

$$D_{[0]}\hat{\sigma} \doteq \hat{\sigma}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) \doteq \frac{1}{2\mathsf{R}}\hat{\sigma}. \tag{5.3.7b}$$

We can solve for both $\hat{\rho}$, and $\hat{\sigma}$ by choosing the perturbation at $\mathsf{v} = 0$ as $\hat{\rho} \doteq \hat{\sigma} \doteq 0$ resembling the unperturbed horizon. Thereby, we have the evolution of the expansion and shear,

$$\hat{\rho}(\mathsf{v}) \doteq \hat{\sigma}(\mathsf{v}) \doteq 0. \tag{5.3.8}$$

Consequently, the perturbed horizon is still characterized by a surface of vanishing expansion and shear.

Additionally, the relations in Eq. (5.2.11) for the perturbations of the transversal expansion ($\hat{\mu} \doteq 0$) and shear simplify using the spin operator with $\hat{\pi} : \mathfrak{s} = -1$ from Eq. (2.3.18),

$$D_{[0]}\hat{\mu} \doteq \eth_{[0]}\hat{\pi} + \hat{\Psi}_2 \doteq 0, \tag{5.3.9a}$$

$$D_{[0]}\hat{\lambda} \doteq -\frac{1}{2\mathsf{R}}\hat{\lambda} + \bar{\eth}_{[0]}\hat{\pi}. \tag{5.3.9b}$$

Hence, we have the differential equation satisfied by $\hat{\pi}$ on the horizon, which can be solved by manifesting $\hat{\Psi}_2$ in terms of the spherical harmonics, and using the properties of the spin operator,

$$\eth_{[0]}\hat{\pi} \doteq -\hat{\Psi}_2. \tag{5.3.10}$$

To manifest the holomorphic structure on the cross-section, we shall expand $\hat{\Psi}_2$ in terms of the (spin weight $\mathfrak{s} = 0$) spherical harmonics,

$$\hat{\Psi}_2 \doteq \sum_{\ell,m} \hat{k}_{\ell,m} Y_{\ell,m}(\xi, \bar{\xi}) \tag{5.3.11}$$

where $\hat{k}_{\ell,m}$ are the perturbations of the source multipole moments, given as $\hat{k}_{\ell,m} = \hat{e}_{\ell,m} + i\hat{b}_{\ell,m}$, which are related to the electric and magnetic moments of the external field. Additionally, we can use Eq. (5.3.4) to relate to the perturbations,

$$\bar{\eth}_{[0]}\eth_{[0]}\hat{P} + \frac{1}{\mathsf{R}^2}\hat{P} + \sum_{\ell,m} \hat{e}_{\ell,m} Y_{\ell,m}(\xi, \bar{\xi}) \doteq 0, \tag{5.3.12}$$

where we realise that $\Delta_{[0]}\hat{P} \doteq \frac{2P_{[0]}^2}{\mathsf{R}^2}\frac{\partial^2}{\partial\xi\partial\bar{\xi}}\hat{P} \doteq 2\left|\delta_{[0]}\right|^2\hat{P} \doteq 2\bar{\eth}_{[0]}\eth_{[0]}\hat{P}$ since $\hat{P}$ has spin-weight $\mathfrak{s} = 0$.

Further, we shall expand $\hat{P}$ in terms of its multipoles using Eq. (2.4.21), to have,

$$\sum_{\ell,m} \hat{c}^{P}_{\ell,m} \bar{\eth}_{[0]} \eth_{[0]} Y_{\ell,m}(\xi,\bar{\xi}) + \sum_{\ell,m} \left(\frac{1}{\mathsf{R}^2} \hat{c}^{P}_{\ell,m} + \hat{e}_{\ell,m} \right) Y_{\ell,m}(\xi,\bar{\xi}) \doteq 0, \tag{5.3.13}$$

where we shall use the normalized[1] eigenvalue relation for the spherical harmonics, in Eq. (2.4.23) such that $\bar{\eth}_{[0]}\eth_{[0]} Y_{\ell,m} \doteq -\frac{\ell(\ell+1)}{2\mathsf{R}^2} Y_{\ell,m}$, to arrive at the multipoles of $\hat{P}$,

$$\sum_{\ell,m} \left(-\frac{1}{2\mathsf{R}^2} \left(\ell^2 + \ell - 2 \right) \hat{c}^{P}_{\ell,m} + \hat{e}_{\ell,m} \right) Y_{\ell,m}(\xi,\bar{\xi}) \doteq 0 \tag{5.3.14}$$

such that, we have the perturbations sourced by the electric multipoles of $\hat{\Psi}_2$,

$$\hat{P}(\xi,\bar{\xi}) \doteq 2\mathsf{R}^2 \sum_{\ell \geq 2,m} \frac{1}{(\ell+2)(\ell-1)} \hat{e}_{\ell,m} Y_{\ell,m}(\xi,\bar{\xi}). \tag{5.3.15}$$

Thereby, we have the differential relation for $\hat{\pi}$ in terms of the spin-weighted spherical harmonics in Eq. (2.4.21),

$$\eth_{[0]}\hat{\pi} \doteq \sum_{\ell,m} \hat{c}^{\pi}_{\ell,m} \eth_{[0]\,-1} Y_{\ell,m}(\xi,\bar{\xi}) \doteq -\sum_{\ell,m} \hat{k}_{\ell,m} Y_{\ell,m}(\xi,\bar{\xi}), \tag{5.3.16}$$

such that we can solve for $\hat{\pi}$ in series, and by using the properties of the spin-weighted spherical harmonics in Eq. (2.4.23) as $\eth_{[0]\,-1} Y_{\ell,m} = \frac{1}{\mathsf{R}} \sqrt{\frac{\ell(\ell+1)}{2}} Y_{\ell,m}$. Comparing the multipole expansions,

$$\hat{\pi} \doteq -\mathsf{R} \sum_{\ell \geq 1,m} \sqrt{\frac{2}{\ell(\ell+1)}} \hat{k}_{\ell,m} \,{}_{-1}Y_{\ell,m}(\xi,\bar{\xi}). \tag{5.3.17}$$

Thereby, we can substitute the form of $\hat{\pi}$ in Eq. (5.3.9) for the transversal shear, to arrive a the spin derivative,

$$\bar{\eth}_{[0]}\hat{\pi} \doteq -\sum_{\ell \geq 2,m} \sqrt{\frac{(\ell+2)(\ell-1)}{\ell(\ell+1)}} \hat{k}_{\ell,m} \,{}_{-2}Y_{\ell,m}(\xi,\bar{\xi}), \tag{5.3.18}$$

where we have used Eq. (2.4.23) to have $\bar{\eth}_{[0]\,-1} Y_{\ell,m} = \frac{1}{\mathsf{R}} \sqrt{\frac{(\ell+2)(\ell-1)}{2}} \,{}_{-2}Y_{\ell,m}$.

Hence, the analagous differential equation for $\hat{\lambda}$ is expanded in terms of the spin-weighted spherical harmonics from Eq. (2.4.23) with $\mathfrak{s} = -2$,

$$\sum_{\ell \geq 2,m} \left(D_{[0]} \hat{c}^{\lambda}_{\ell,m} + \frac{1}{2\mathsf{R}} \hat{c}^{\lambda}_{\ell,m} - \sqrt{\frac{(\ell+2)(\ell-1)}{\ell(\ell+1)}} \hat{k}_{\ell,m} \right) {}_{-2}Y_{\ell,m}(\xi,\bar{\xi}) \doteq 0. \tag{5.3.19}$$

[1] The spherical harmonics are subjected to normalization by R to lie on the unit-sphere.

Similar to the expression for the transversal shear in Eq. (4.1.24), we arrive at the solution using $\hat{\lambda}(0) \doteq 0$ at $\mathsf{v} = 0$,

$$\hat{\lambda} \doteq 2\mathsf{R}\left[1 - \exp\left(-\frac{\mathsf{v}}{2\mathsf{R}}\right)\right] \sum_{\ell \geq 2, m} \sqrt{\frac{(\ell+2)(\ell-1)}{\ell(\ell+1)}} \hat{k}_{\ell,m\,-2} Y_{\ell,m}(\xi, \bar{\xi}) \tag{5.3.20}$$

Further, we have the angular field equations from Eq. (5.2.11) on the horizon, in linear order perturbations,

$$\bar{\delta}_{[0]}\hat{\sigma} \doteq \hat{\Psi}_1 \doteq 0, \tag{5.3.21a}$$

$$\begin{aligned} \delta_{[0]}\hat{\lambda} &\doteq \mu_{[0]}\hat{\pi} + \hat{\lambda}\left(\bar{\alpha}_{[0]} - 3\beta_{[0]}\right) - \hat{\Psi}_3 \\ &\doteq \frac{\sqrt{2}\xi}{\mathsf{R}}\hat{\lambda} - \frac{1}{\mathsf{R}}\hat{\pi} - \hat{\Psi}_3, \end{aligned} \tag{5.3.21b}$$

where we used the relations $\hat{\mu} \doteq \hat{\bar{\mu}} \doteq 0$, since μ is real from Eq. (4.1.8), and have assumed $\hat{\bar{\delta}}\mu_{[0]} \doteq \hat{P}\delta_{[0]}\mu_{[0]} \doteq 0$ by spherical symmetry.

Clearly $\hat{\Psi}_1 \doteq 0$, and we have the perturbations of the field tensor $\hat{\Psi}_3$ which depends on $\hat{\lambda}$,

$$\hat{\Psi}_3 \doteq -\eth_{[0]}\hat{\lambda} - \frac{1}{\mathsf{R}}\hat{\pi}, \tag{5.3.22}$$

using the spin operator for $\hat{\lambda}$ with $\mathfrak{s} = -2$.

Now, we can rewrite the temporal Bianchi identities from Eq. (5.2.12) by using the spin-operators for the Weyl tensors to arrive at,

$$D_{[0]}\hat{\Psi}_1 \doteq 2\epsilon_{[0]}\hat{\Psi}_1 \doteq 0, \tag{5.3.23a}$$

$$D_{[0]}\hat{\Psi}_2 \doteq \bar{\delta}_{[0]}\hat{\Psi}_1 - 2\alpha_{[0]}\hat{\Psi}_1 \doteq \bar{\eth}_{[0]}\hat{\Psi}_1 \doteq 0. \tag{5.3.23b}$$

For the perturbed spin coefficients $\hat{\alpha}$, $\hat{\bar{\beta}}$, determining the connection, we have Eq. (5.2.11) simplifying using the above relations,

$$D_{[0]}\hat{\alpha} \doteq \bar{\epsilon}_{[0]}\left(\hat{\pi} - \hat{\alpha} - \hat{\bar{\beta}}\right) \doteq 0, \tag{5.3.24a}$$

$$D_{[0]}\hat{\bar{\beta}} \doteq \epsilon_{[0]}\left(\hat{\bar{\pi}} - \hat{\bar{\alpha}} - \hat{\beta}\right) \doteq 0, \tag{5.3.24b}$$

where we have additionally assumed $\hat{\delta}\epsilon_{[0]} = \hat{P}\delta_{[0]}\epsilon_{[0]} = 0$ by the nature of the intrinsic horizon to be spherically symmetric.

Thereby, we have the relations for $\hat{\pi} = \hat{\alpha} + \hat{\bar{\beta}}$ and the connection $\hat{a}$, given by,

$$D_{[0]}\hat{\pi} \doteq D_{[0]}\hat{a} \doteq 0. \tag{5.3.25}$$

The rest of the temporal Bianchi identities in Eq. (5.2.12) simplify as follows,

$$D_{[0]}\hat{\Psi}_3 \doteq \bar{\delta}_{[0]}\hat{\Psi}_2 + 3\hat{\pi}\Psi_2^{[0]} - 2\epsilon_{[0]}\hat{\Psi}_3$$

$$\doteq \bar{\eth}_{[0]}\hat{\Psi}_2 - \frac{1}{2\mathsf{R}}\hat{\Psi}_3 - \frac{3}{2\mathsf{R}^2}\hat{\pi}, \tag{5.3.26a}$$

$$D_{[0]}\hat{\Psi}_4 \doteq \bar{\delta}_{[0]}\hat{\Psi}_3 - 3\hat{\lambda}\Psi_2^{[0]} + 2\alpha_{[0]}\hat{\Psi}_3 - 4\epsilon_{[0]}\hat{\Psi}_4$$

$$\doteq \bar{\eth}_{[0]}\hat{\Psi}_3 - \frac{1}{\mathsf{R}}\hat{\Psi}_4 + \frac{3}{2\mathsf{R}^2}\hat{\lambda}, \tag{5.3.26b}$$

where we have assumed spherical symmetry for $\Psi_2^{[0]}$ such that $\hat{\bar{\delta}}_{[0]}\Psi_2^{[0]} \doteq \hat{P}\bar{\delta}_{[0]}\Psi_2^{[0]} \doteq 0$.

Analogous to before, we arrive at the differential equation for $\hat{\Psi}_3$ in terms of the spin-weighted spherical harmonics,

$$\sum_{\ell\geq 1,m}\left(D_{[0]}\hat{c}^{\Psi_3}_{\ell,m} + \frac{1}{2\mathsf{R}}\hat{c}^{\Psi_3}_{\ell,m}\right){}_{-1}Y_{\ell,m}(\xi,\bar{\xi}) \doteq \frac{3}{\sqrt{2}\mathsf{R}}\sum_{\ell\geq 1,m}\sqrt{\frac{1}{\ell(\ell+1)}}\hat{k}_{\ell,m}{}_{-1}Y_{\ell,m}(\xi,\bar{\xi}) + \sum_{\ell,m}\hat{k}_{\ell,m}\bar{\eth}_{[0]}Y_{\ell,m}(\xi,\bar{\xi}), \tag{5.3.27}$$

such that, we have the relation,

$$\sum_{\ell\geq 1,m}\left(D_{[0]}\hat{c}^{\Psi_3}_{\ell,m} + \frac{1}{2\mathsf{R}}\hat{c}^{\Psi_3}_{\ell,m} + \frac{1}{\sqrt{2}\mathsf{R}}\frac{1}{\sqrt{\ell(\ell+1)}}\left[\ell(\ell+1)-3\right]\hat{k}_{\ell,m}\right){}_{-1}Y_{\ell,m}(\xi,\bar{\xi}) \doteq 0, \tag{5.3.28}$$

where Eq. (2.4.23) is used to derive $\bar{\eth}_{[0]}Y_{\ell,m} = \frac{1}{\mathsf{R}}\sqrt{\frac{\ell(\ell+1)}{2}}{}_{-1}Y_{\ell,m}$.

The solution to the equation is thereby analogous to the transversal shear in Eq. (5.3.20), by using the initial condition $\hat{\Psi}_3(0) \doteq 0$ at $\mathsf{v} = 0$,

$$\hat{\Psi}_3 \doteq -\sqrt{2}\left[1-\exp\left(-\frac{\mathsf{v}}{2\mathsf{R}}\right)\right]\sum_{\ell\geq 1,m}\frac{1}{\sqrt{\ell(\ell+1)}}\left[\ell(\ell+1)-3\right]\hat{k}_{\ell,m}{}_{-1}Y_{\ell,m}(\xi,\bar{\xi}) \tag{5.3.29}$$

Finally, we delve into perturbations of the radiative Weyl tensor $\hat{\Psi}_4$, expanding in terms of spin-weighted spherical harmonics with $\mathfrak{s} = 2$,

$$\sum_{\ell\geq 2,m}\left(D_{[0]}\hat{c}^{\Psi_4}_{\ell,m} + \frac{1}{\mathsf{R}}\hat{c}^{\Psi_4}_{\ell,m}\right){}_{-2}Y_{\ell,m}(\xi,\bar{\xi}) \doteq \sum_{\ell\geq 1,m}\hat{c}^{\Psi_3}_{\ell,m}\bar{\eth}_{[0]}{}_{-1}Y_{\ell,m}(\xi,\bar{\xi}) + \frac{3}{2\mathsf{R}^2}\sum_{\ell\geq 2,m}\hat{c}^{\lambda}_{\ell,m}{}_{-2}Y_{\ell,m}(\xi,\bar{\xi}), \tag{5.3.30}$$

where we have used the relations in Eq. (5.3.29) and Eq. (5.3.20).

Further, using the derivative in Eq. (2.4.23) is used to derive $\bar{\eth}_{[0]}{}_{-1}Y_{\ell,m} = \frac{1}{\mathsf{R}}\sqrt{\frac{(\ell+2)(\ell+1)}{2}}{}_{-2}Y_{\ell,m}$, we have,

$$\sum_{\ell\geq 2,m}\left(D_{[0]}\hat{c}^{\Psi_4}_{\ell,m} + \frac{1}{\mathsf{R}}\hat{c}^{\Psi_4}_{\ell,m}\right){}_{-2}Y_{\ell,m}(\xi,\bar{\xi}) \doteq \sum_{\ell\geq 2,m}\frac{1}{\mathsf{R}}\left[1-\exp\left(-\frac{\mathsf{v}}{2\mathsf{R}}\right)\right]$$

$$\times\left(3-\ell(\ell+1)+3\sqrt{\frac{(\ell+2)(\ell-1)}{\ell(\ell+1)}}\right)\hat{k}_{\ell,m\ -2}Y_{\ell,m}(\xi,\bar{\xi}). \tag{5.3.31}$$

Assuming at initial time $\mathsf{v}=0$, we have $\hat{\Psi}_4(0)\doteq 0$, thereby, we have the solution for the perturbation,

$$\hat{\Psi}_4 \doteq \sum_{\ell\geq 2,m}\left[1-\exp\left(-\frac{\mathsf{v}}{2\mathsf{R}}\right)\right]^2\left(3-\ell(\ell+1)+3\sqrt{\frac{(\ell+2)(\ell-1)}{\ell(\ell+1)}}\right)_{-2}\hat{k}_{\ell,m\ -2}Y_{\ell,m}(\xi,\bar{\xi}) \tag{5.3.32}$$

Unlike $\hat{\lambda}$ and $\hat{\Psi}_3$, $\hat{\Psi}_4$ is foliation independent, which can be manifested by realising the second time derivative of $\hat{\Psi}_4$ from Eq. (5.3.26),

$$\begin{aligned} D^2_{[0]}\hat{\Psi}_4 &\doteq D_{[0]}\left(\bar{\eth}_{[0]}\hat{\Psi}_3\right)-\frac{1}{\mathsf{R}}D_{[0]}\hat{\Psi}_4+\frac{3}{2\mathsf{R}^2}D_{[0]}\hat{\lambda} \\ &\doteq \bar{\eth}_{[0]}\left(D_{[0]}\hat{\Psi}_3\right)-\frac{1}{\mathsf{R}}D_{[0]}\hat{\Psi}_4+\frac{3}{2\mathsf{R}^2}D_{[0]}\hat{\lambda} \end{aligned} \tag{5.3.33}$$

since the angular operator $\bar{\eth}_{[0]}$ is independent of the temporal coordinate on the horizon, and thus D commutes with $\bar{\eth}_{[0]}$.

Further, substituting $D_{[0]}\hat{\Psi}_3$ from Eq. (5.3.26), and $D_{[0]}\lambda$ from Eq. (5.3.9), we have,

$$\begin{aligned} D^2_{[0]}\hat{\Psi}_4 &\doteq \bar{\eth}_{[0]}\left(\bar{\eth}_{[0]}\hat{\Psi}_2-\frac{1}{2\mathsf{R}}\hat{\Psi}_3-\frac{3}{2\mathsf{R}^2}\hat{\pi}\right)-\frac{1}{\mathsf{R}}D_{[0]}\hat{\Psi}_4+\frac{3}{2\mathsf{R}^2}\left(-\frac{1}{2\mathsf{R}}\hat{\lambda}+\bar{\eth}_{[0]}\hat{\pi}\right) \\ &\doteq \bar{\eth}^2_{[0]}\hat{\Psi}_2-\frac{1}{2\mathsf{R}}\bar{\eth}_{[0]}\hat{\Psi}_3-\frac{1}{\mathsf{R}}D_{[0]}\hat{\Psi}_4-\frac{3}{4\mathsf{R}^3}\hat{\lambda}, \end{aligned} \tag{5.3.34}$$

Now, we note the expression for $\bar{\eth}_{[0]}\hat{\Psi}_3$ from Eq. (5.3.26), to have,

$$\begin{aligned} D^2_{[0]}\hat{\Psi}_4 &\doteq \bar{\eth}^2_{[0]}\hat{\Psi}_2-\frac{1}{2\mathsf{R}}\left(D_{[0]}\hat{\Psi}_4+\frac{1}{\mathsf{R}}\hat{\Psi}_4-\frac{3}{2\mathsf{R}^2}\hat{\lambda}\right)-\frac{1}{\mathsf{R}}D_{[0]}\hat{\Psi}_4-\frac{3}{4\mathsf{R}^3}\hat{\lambda} \\ &\doteq \bar{\eth}^2_{[0]}\hat{\Psi}_2-\frac{3}{2\mathsf{R}}D_{[0]}\hat{\Psi}_4-\frac{1}{2\mathsf{R}^2}\hat{\Psi}_4 \end{aligned} \tag{5.3.35}$$

Since $\hat{\Psi}_2$ is time-independent on the horizon with $D_{[0]}\hat{\Psi}_2\doteq 0$ from Eq. (5.3.23), we have the foliation independence of the evolution of the radiative $\hat{\Psi}_4$, and it is intrinsically determined by the curvature. Thereby, we shall use the variable separation, $\hat{\Psi}_4\doteq\mathcal{T}(\mathsf{v})\mathcal{Y}(\xi,\bar{\xi})$, such that we obtain the independent differential equations,

$$D^2_{[0]}\mathcal{T}+\frac{3}{2\mathsf{R}}D_{[0]}\mathcal{T}+\frac{1}{2\mathsf{R}^2}\mathcal{T}\doteq K, \tag{5.3.36a}$$

$$K\mathcal{Y}\doteq\bar{\eth}^2_{[0]}\hat{\Psi}_2, \tag{5.3.36b}$$

where K is the separation constant.

Therefore, the angular dependence of $\hat{\Psi}_4$ can only be freely specified at the horizon when $K \doteq 0$. Hence, we have a restriction on the curvature perturbation to have $\bar{\eth}^2_{[0]}\hat{\Psi}_2 \doteq 0$, such that $\hat{\Psi}_2$ can only be monopolar or dipolar, since the angular derivatives restrict terms of higher spherical moments.

5.3.2 Near Horizon Perturbations

Substituting the unperturbed radial evolutions, the radial field equations from Eq. (5.2.13) for the expansion μ,

$$\Delta\hat{\mu} = -2\mu_{[0]}\hat{\mu} = \frac{2}{\mathrm{r}}\hat{\mu}, \tag{5.3.37a}$$

which we ensure is unaffected by the perturbation on the horizon, such that $\hat{\mu} \doteq 0$, hence, we have,

$$\hat{\mu} = 0. \tag{5.3.37b}$$

Further, to solve for the other spin coefficients and the Weyl scalars, we need to specify the nature of the radiative scalar, $\hat{\Psi}_4$.

The radial evolution for $\hat{\Psi}_4$ is specified implicitly through the temporal equation $D\hat{\Psi}_4$ in Eq. (4.1.17), since the directional derivative D contains a radial derivative away from the horizon by definition, from the frame functions in Eq. (4.2.11). Thereby, the temporal, radial, and angular dependence of $\hat{\Psi}_4$ is non-trivial.

The evolution equation in Eq. (4.1.17) for $\hat{\Psi}_4$ simplifies using the spin-operator as,

$$\begin{aligned} D_{[0]}\hat{\Psi}_4 &= \bar{\eth}_{[0]}\hat{\Psi}_3 - 3\Psi_2^{[0]}\hat{\lambda} - \hat{\Psi}_4\left(4\epsilon_{[0]} - \varrho_{[0]}\right) \\ &= \bar{\delta}_{[0]}\hat{\Psi}_3 - a_{[0]}\hat{\Psi}_3 - 3\Psi_2^{[0]}\hat{\lambda} - \hat{\Psi}_4\left(4\epsilon_{[0]} - \varrho_{[0]}\right). \end{aligned} \tag{5.3.38}$$

The radial derivative of the evolution equation for $\hat{\Psi}_4$ in the field equation, Eq. (4.1.17), is

$$\begin{aligned} \Delta D_{[0]}\hat{\Psi}_4 &= \Delta\left(\bar{\delta}_{[0]}\hat{\Psi}_3\right) - \Delta\left(a_{[0]}\hat{\Psi}_3\right) - 3\Delta\left(\Psi_2^{[0]}\hat{\lambda}\right) - \Delta\left(\hat{\Psi}_4\left(4\epsilon_{[0]} - \varrho_{[0]}\right)\right) \\ &= \bar{\delta}_{[0]}\left(\Delta\hat{\Psi}_3\right) - \mu_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_3 - \Delta a_{[0]}\hat{\Psi}_3 - a_{[0]}\Delta\hat{\Psi}_3 - 3\Delta\Psi_2^{[0]}\hat{\lambda} - 3\Psi_2^{[0]}\Delta\hat{\lambda} \\ &\quad - \hat{\Psi}_4\Delta\left(4\epsilon_{[0]} - \varrho_{[0]}\right) - \Delta\hat{\Psi}_4\left(4\epsilon_{[0]} - \varrho_{[0]}\right). \end{aligned} \tag{5.3.39}$$

where we note that $\bar{\delta}_{[0]}$ and Δ do not commute, with $[\bar{\delta}_{[0]}, \Delta] = \mu_{[0]}\bar{\delta}_{[0]}$, from the commutation relations in Eq. (4.1.6) for the spherically symmetric background spacetime.

We substitute the radial differential equation in Eq. (5.2.14) for $\hat{\Psi}_3$, by computing

$$\begin{aligned}\bar{\delta}_{[0]}\left(\Delta\hat{\Psi}_3\right) &= \bar{\delta}_{[0]}\left(\bar{\eth}_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\hat{\Psi}_3\right) = \bar{\delta}_{[0]}\left(\delta_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\hat{\Psi}_3\right)\\ &= \bar{\delta}_{[0]}\delta_{[0]}\hat{\Psi}_4 - 2\bar{\delta}_{[0]}\bar{a}_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_3. \qquad (5.3.40)\end{aligned}$$

Hence, the radial derivative of the evolution equation simplifies,

$$\begin{aligned}\Delta D_{[0]}\hat{\Psi}_4 = {} & \bar{\delta}_{[0]}\delta_{[0]}\hat{\Psi}_4 - 2\bar{\delta}_{[0]}\bar{a}_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_4 - 5\mu_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_3 - \Delta a_{[0]}\hat{\Psi}_3\\ & - a_{[0]}\delta_{[0]}\hat{\Psi}_4 + 2\left|a_{[0]}\right|^2\hat{\Psi}_4 + 4a_{[0]}\mu_{[0]}\hat{\Psi}_3 - 3\Delta\Psi_2^{[0]}\hat{\lambda} - 3\Psi_2^{[0]}\Delta\hat{\lambda}\\ & - \hat{\Psi}_4\Delta\left(4\epsilon_{[0]} - \rho_{[0]}\right) - \Delta\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right). \qquad (5.3.41)\end{aligned}$$

Thereby, we now substitute for $\bar{\eth}_{[0]}\hat{\Psi}_3$ from Eq. (4.1.17) with $\bar{\delta}_{[0]}\hat{\Psi}_3 = D_{[0]}\hat{\Psi}_4 + a_{[0]}\hat{\Psi}_3 + 3\Psi_2^{[0]}\hat{\lambda} + \hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right)$, to have,

$$\begin{aligned}\Delta D_{[0]}\hat{\Psi}_4 = {} & \bar{\delta}_{[0]}\delta_{[0]}\hat{\Psi}_4 - 2\bar{\delta}_{[0]}\bar{a}_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_4 - a_{[0]}\delta_{[0]}\hat{\Psi}_4 + 2\left|a_{[0]}\right|^2\hat{\Psi}_4\\ & - 5\mu_{[0]}D_{[0]}\hat{\Psi}_4 - \left(\mu_{[0]}a_{[0]} + \Delta a_{[0]}\right)\hat{\Psi}_3 - 5\mu_{[0]}\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right)\\ & - 15\mu_{[0]}\Psi_2^{[0]}\hat{\lambda} - 3\Delta\Psi_2^{[0]}\hat{\lambda} - 3\Psi_2^{[0]}\Delta\hat{\lambda} - \hat{\Psi}_4\Delta\left(4\epsilon_{[0]} - \rho_{[0]}\right)\\ & - \Delta\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right). \qquad (5.3.42)\end{aligned}$$

Finally, we substitute the radial field equation from Eq. (5.2.13) for $\hat{\lambda}$ as $\Delta\hat{\lambda} = -2\mu_{[0]}\hat{\lambda} - \hat{\Psi}_4$, to arrive at,

$$\begin{aligned}\Delta D_{[0]}\hat{\Psi}_4 = {} & \bar{\delta}_{[0]}\delta_{[0]}\hat{\Psi}_4 - 2\bar{\delta}_{[0]}\bar{a}_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_4 - a_{[0]}\delta_{[0]}\hat{\Psi}_4 + 2\left|a_{[0]}\right|^2\hat{\Psi}_4\\ & - 5\mu_{[0]}D_{[0]}\hat{\Psi}_4 - \left(\mu_{[0]}a_{[0]} + \Delta a_{[0]}\right)\hat{\Psi}_3 - 5\mu_{[0]}\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right)\\ & - 3\left(\Delta\Psi_2^{[0]} + 3\mu_{[0]}\Psi_2^{[0]}\right)\hat{\lambda} + 3\Psi_2^{[0]}\hat{\Psi}_4 - \hat{\Psi}_4\Delta\left(4\epsilon_{[0]} - \rho_{[0]}\right)\\ & - \Delta\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right). \qquad (5.3.43)\end{aligned}$$

For the spherically symmetric background, we can further use the radial evolution equation for $a_{[0]}$ from Eq. (4.1.16), as $\Delta a_{[0]} = -\mu_{[0]}a_{[0]}$, and the radial Bianchi identity for $\Psi_2^{[0]}$ from Eq. (4.1.17) as $\Delta\Psi_2^{[0]} = -3\mu_{[0]}\Psi_2^{[0]}$, to simplify, as

$$\begin{aligned}\Delta D_{[0]}\hat{\Psi}_4 = {} & \bar{\delta}_{[0]}\delta_{[0]}\hat{\Psi}_4 - 2\bar{\delta}_{[0]}\bar{a}_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_4 - a_{[0]}\delta_{[0]}\hat{\Psi}_4 + 2\left|a_{[0]}\right|^2\hat{\Psi}_4\\ & - 5\mu_{[0]}D_{[0]}\hat{\Psi}_4 - 5\mu_{[0]}\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right) + 3\Psi_2^{[0]}\hat{\Psi}_4 - \hat{\Psi}_4\Delta\left(4\epsilon_{[0]} - \rho_{[0]}\right)\\ & - \Delta\hat{\Psi}_4\left(4\epsilon_{[0]} - \rho_{[0]}\right). \qquad (5.3.44)\end{aligned}$$

For $\hat{\Psi}_4$ with spin-weight $\mathfrak{s} = -2$, we realize the action of the spin-operator,

$$\begin{aligned}\bar{\eth}_{[0]}\eth_{[0]}\hat{\Psi}_4 &= \bar{\eth}_{[0]}\left(\delta_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\hat{\Psi}_4\right)\\ &= \bar{\delta}_{[0]}\left(\delta_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\hat{\Psi}_4\right) - a_{[0]}\left(\delta_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\hat{\Psi}_4\right)\\ &= \bar{\delta}_{[0]}\delta_{[0]}\hat{\Psi}_4 - 2\bar{\delta}_{[0]}\bar{a}_{[0]}\hat{\Psi}_4 - 2\bar{a}_{[0]}\bar{\delta}_{[0]}\hat{\Psi}_4 - a_{[0]}\delta_{[0]}\hat{\Psi}_4 + 2\left|a_{[0]}\right|^2\hat{\Psi}_4,\end{aligned} \tag{5.3.45}$$

where we have expanded the spin-operator acting on $\eth_{[0]}\hat{\Psi}_4$, which has spin-weight $\mathfrak{s} = -1$.

Thereby, we obtain the *Teukolsky equation*, as a wave equation for the radiative Weyl scalar, $\hat{\Psi}_4$,

$$\mathcal{O}^{\mathsf{T}}_{[0]}\hat{\Psi}_4 = 0, \tag{5.3.46a}$$

where the differential operator is given by, which can be substituted as,

$$\begin{aligned}\mathcal{O}^{\mathsf{T}}_{[0]} &= \Delta D_{[0]} - \bar{\eth}_{[0]}\eth_{[0]} + 5\mu_{[0]}D_{[0]} + 5\mu_{[0]}\left(4\epsilon_{[0]} - \rho_{[0]}\right) + \Delta\left(4\epsilon_{[0]} - \rho_{[0]}\right)\\ &\quad + \left(4\epsilon_{[0]} - \rho_{[0]}\right)\Delta - 3\Psi_2^{[0]}.\end{aligned} \tag{5.3.46b}$$

For the Schwarzschild case, we can substitute the unperturbed spin coefficients and Weyl scalars in Eq. (5.3.1), and can be expanded in terms of the directional derivatives of the unperturbed basis vectors in Eq. (4.2.11), as

$$\mathcal{O}^{\mathsf{T}}_{[0]} = \left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\partial_{\mathsf{r}}^2 - \left(\partial_{\mathsf{r}} + \frac{5}{\mathsf{r}}\right)\partial_{\mathsf{v}} - \bar{\eth}_{[0]}\eth_{[0]} + \frac{9}{2\mathsf{r}}\left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\partial_{\mathsf{r}} - \frac{2}{\mathsf{r}^2}. \tag{5.3.47}$$

We use the variable separation extending from the horizon in Eq. (5.3.36), with $\Psi_4 = \mathcal{T}(\mathsf{v})\mathcal{Y}(\xi,\bar{\xi})\mathcal{R}(\mathsf{r})$ where $\mathcal{R}(\mathsf{R}) = 1$ on the horizon. Thereby, we arrive at the differential equation,

$$\begin{aligned}\mathcal{O}^{\mathsf{T}}_{[0]} &= \left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\frac{1}{\mathcal{R}}\partial_{\mathsf{r}}^2\mathcal{R} - \frac{1}{\mathcal{T}}\partial_{\mathsf{v}}\mathcal{T}\frac{1}{\mathcal{R}}\partial_{\mathsf{r}}\mathcal{R} - \frac{5}{\mathsf{r}}\frac{1}{\mathcal{T}}\partial_{\mathsf{v}}\mathcal{T}\\ &\quad + \frac{9}{2\mathsf{r}}\left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\frac{1}{\mathcal{R}}\partial_{\mathsf{r}}\mathcal{R} - \frac{2}{\mathsf{r}^2} - \frac{1}{\mathcal{Y}}\bar{\eth}_{[0]}\eth_{[0]}\mathcal{Y},\end{aligned} \tag{5.3.48a}$$

such that, using the sepration constants k and χ, we have,

$$\frac{1}{\mathcal{T}}\partial_{\mathsf{v}}\mathcal{T} = -\chi, \tag{5.3.48b}$$

$$\bar{\eth}_{[0]}\eth_{[0]}\mathcal{Y} = -k\mathcal{Y}. \tag{5.3.48c}$$

Thereby, using the above independent differential equations, the radial function must satisfy,

$$\left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\partial_{\mathsf{r}}^2\mathcal{R} + \left(\chi + \frac{9}{2\mathsf{r}}\left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\right)\partial_{\mathsf{r}}\mathcal{R} + \left(k + \frac{5}{\mathsf{r}}\chi - \frac{2}{\mathsf{r}^2}\right)\mathcal{R} = 0. \tag{5.3.48d}$$

Further, we also need to satisfy the boundary conditions at the horizon given by Eq. (5.3.36). Thereby, on the horizon, we have,

$$\begin{aligned} K &\doteq D_{[0]}^2\mathcal{T} + \frac{3}{2\mathsf{R}}D_{[0]}\mathcal{T} + \frac{1}{2\mathsf{R}^2}\mathcal{T} \doteq \partial_{\mathsf{v}}^2\mathcal{T} + \frac{3}{2\mathsf{R}}\partial_{\mathsf{v}}\mathcal{T} + \frac{1}{2\mathsf{R}^2}\mathcal{T} \\ &\doteq \left(\chi^2 - \frac{3}{2\mathsf{R}}\chi + \frac{1}{2\mathsf{R}^2}\right)\mathcal{T} \doteq \left(\chi - \frac{1}{2\mathsf{R}}\right)\left(\chi - \frac{1}{\mathsf{R}}\right)\mathcal{T}, \end{aligned} \tag{5.3.49}$$

which has two solutions, $\chi = \frac{1}{\mathsf{R}}, \frac{1}{2\mathsf{R}}$ with $K = 0$ and $\chi = 0$ with $K = \frac{1}{2\mathsf{R}^2}\mathcal{T}$.

Angular Independence

When $K = 0$, we have $\bar{\eth}_{[0]}^2\hat{\Psi}_2 \doteq 0$, and the angular dependence of $\hat{\Psi}_4$ can be freely specified, independent of the angular behaviour of $\hat{\Psi}_2$. We have the angular function $\mathcal{Y}$ spanned in $\mathfrak{s} = -2$ spherical harmonics ${}_{-2}Y_{\ell,m}(\xi,\bar{\xi})$, hence we set $k = \frac{1}{\mathsf{r}^2}\frac{(\ell+2)(\ell-1)}{2}$ for the angular equation[2].

The radial equation can be solved in terms of confluent Heun functions as elaborated in Sec. A.2. The confluent Heun function is the standard solution regular at $\mathsf{r} = \mathsf{R}$ with appropriate normalization, which can be written as a linear combination of two independent confluent Heun solutions for $\chi = \frac{1}{\mathsf{R}}, \frac{1}{2\mathsf{R}}$, with the time dependence $\mathcal{T}(\mathsf{v}) = \exp(-\chi\mathsf{v})$. Thereby, the general form of the perturbed Weyl scalar $\hat{\Psi}_4$, is,

$$\hat{\Psi}_4 = \sum_{\ell\geq 2,m}\sum_{n=1,2}\hat{y}_{\ell m}b_n H_\ell^n(\mathsf{z})\exp\left(-\frac{n\mathsf{v}}{2\mathsf{R}}\right){}_{-2}Y_{\ell,m}(\xi,\bar{\xi}). \tag{5.3.50}$$

Time Independence

When $\chi = 0$, we have the time independent solution for $\hat{\Psi}_4$, with the temporal function $\mathcal{T} = 2\mathsf{R}^2K$. The angular part of $\hat{\Psi}_4$ is given by the $\mathcal{Y}(\xi,\bar{\xi}) = 2\mathsf{R}^2\bar{\eth}_{[0]}^2\hat{\Psi}_2$, which solves the angular part of the Teukolsky equation, with the eigenvalue $k = \frac{1}{\mathsf{r}^2}\frac{(\ell+2)(\ell-1)}{2}$. Thereby, the radial evolution in Eq. (5.3.48) simplifies as,

$$2\mathsf{r}(\mathsf{r}-\mathsf{R})\,\partial_{\mathsf{r}}^2\mathcal{R} + 9(\mathsf{r}-\mathsf{R})\,\partial_{\mathsf{r}}\mathcal{R} + (\ell+3)(\ell-2)\mathcal{R} = 0. \tag{5.3.51}$$

Transforming the variables as detailed in Sec. A.3 and using the (Gauss) hypergeometric function, we have the simplification for the Weyl scalar with appropriate normalization,

$$\hat{\Psi}_4 = \frac{\mathsf{R}^2}{\mathsf{r}^2}\sum_{\ell\geq 2,m}\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)}\,{}_2\mathsf{F}_1\left[\begin{matrix}-\ell,\ \ell+1\\ 3\end{matrix}\,\middle|\,\frac{\mathsf{r}}{\mathsf{R}}-1\right]\hat{k}_{\ell,m}\,{}_{-2}Y_{\ell,m}(\xi,\bar{\xi}). \tag{5.3.52}$$

[2]From the eigenvalue equation for the spherical harmonics in Eq. (2.4.23), we have $\bar{\eth}_{[0]}\eth_{[0]}\,{}_{-2}Y_{\ell,m}(\xi,\bar{\xi}) \doteq -\frac{1}{\mathsf{r}^2}\frac{(\ell+2)(\ell-1)}{2}\,{}_{-2}Y_{\ell,m}(\xi,\bar{\xi})$.

Now, we proceed to solve the rest of the Weyl scalars radially for the time-independent case. From Eq. (5.2.14), we have the evolution of the Weyl scalars, where we substitute the unperturbed spin coefficients and the appropriate Weyl scalars, such that,

$$\begin{aligned}\Delta\hat{\Psi}_3 = -\frac{\partial\hat{\Psi}_3}{\partial \mathsf{r}} &= \delta_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\hat{\Psi}_3 + 4\beta_{[0]}\hat{\Psi}_4 = \eth_{[0]}\hat{\Psi}_4 - 4\mu_{[0]}\hat{\Psi}_3 \\ &= \frac{\mathsf{R}^2}{\mathsf{r}^2}\sum_{\ell\geq 2,m}(\ell+2)\sqrt{\frac{\ell(\ell+1)(\ell-1)(\ell+3)}{2}}\,{}_2\mathsf{F}_1\left[\begin{matrix}-\ell, & \ell+1 \\ & 3\end{matrix}\,\middle|\,\frac{\mathsf{r}}{\mathsf{R}}-1\right] \\ &\qquad \times \hat{k}_{\ell,m-2}Y_{\ell,m}(\xi,\bar{\xi}) + \frac{4}{\mathsf{r}}\hat{\Psi}_3,\end{aligned} \tag{5.3.53a}$$

$$\Delta\hat{\Psi}_2 = -\frac{\partial\hat{\Psi}_3}{\partial \mathsf{r}} = \delta_{[0]}\hat{\Psi}_3 - 3\mu_{[0]}\hat{\Psi}_2 + 2\beta_{[0]}\hat{\Psi}_3 = \eth_{[0]}\hat{\Psi}_3 + \frac{3}{\mathsf{r}}\hat{\Psi}_2, \tag{5.3.53b}$$

$$\Delta\hat{\Psi}_1 = -\frac{\partial\hat{\Psi}_3}{\partial \mathsf{r}} = \delta_{[0]}\hat{\Psi}_2 - 2\mu_{[0]}\hat{\Psi}_1 = \eth_{[0]}\hat{\Psi}_2 + \frac{2}{\mathsf{r}}\hat{\Psi}_1, \tag{5.3.53c}$$

where we have utilized the unperturbed spin coefficients.

Thereby, we can solve sequentially, and substitute the perturbed values at the horizon by setting $\exp\left(-\frac{\mathsf{v}}{2\mathsf{R}}\right) \to 0$ at $\mathsf{r} = \mathsf{R}$, such that, $\hat{\Psi}_3(\mathsf{R}) = -\sqrt{2}\sum_{\ell\geq 1,m}\frac{1}{\sqrt{\ell(\ell+1)}}[\ell(\ell+1)-3]\hat{k}_{\ell,m-1}Y_{\ell,m}(\xi,\bar{\xi})$ from Eq. (5.3.29),

$$\begin{aligned}\hat{\Psi}_3 = \frac{\mathsf{R}}{\sqrt{2}\mathsf{r}}\sum_{\ell,m}\frac{1}{\sqrt{\ell(\ell+1)}}\Bigg({}_2\mathsf{F}_1\left[\ell+3, 2-\ell, 1, \frac{\mathsf{r}}{\mathsf{R}}-1\right] \\ -(\ell+2)(\ell-1)\,{}_2\mathsf{F}_1\left[\ell+3, 2-\ell, 2, \frac{\mathsf{r}}{\mathsf{R}}-1\right]\Bigg)\hat{k}_{\ell,m-1}Y_{\ell,m}(\xi,\bar{\xi}),\end{aligned} \tag{5.3.54}$$

and $\hat{\Psi}_2(\mathsf{R}) = \sum_{\ell,m}\hat{k}_{\ell,m}Y_{\ell,m}(\xi,\bar{\xi})$ from Eq. (5.3.11),

$$\begin{aligned}\hat{\Psi}_2 = \frac{\mathsf{R}}{\mathsf{r}}\sum_{\ell,m}\frac{1}{(\ell+2)}\Bigg((\ell-1)\,{}_2\mathsf{F}_1\left[1-\ell, \ell+2, 1, \frac{\mathsf{r}}{\mathsf{R}}-1\right] \\ +3\,{}_2\mathsf{F}_1\left[2-\ell, \ell+2, 1, \frac{\mathsf{r}}{\mathsf{R}}-1\right]\Bigg)\hat{k}_{\ell,m}Y_{\ell,m}(\xi,\bar{\xi}),\end{aligned} \tag{5.3.55}$$

and $\hat{\Psi}_1(\mathsf{R}) = 0$ from Eq. (5.3.21),

$$\hat{\Psi}_1 = \tag{5.3.56}$$

Further, we can solve the radial equations in Eq. (5.2.13),

$$\Delta\hat{\lambda} = -2\mu_{[0]}\hat{\lambda} - \hat{\Psi}_4 = \frac{2}{\mathsf{r}}\hat{\lambda} - \hat{\Psi}_4, \tag{5.3.57a}$$

$$\Delta\hat{\sigma} = -\mu_{[0]}\hat{\sigma} - \rho_{[0]}\hat{\bar{\lambda}} = \frac{1}{\mathsf{r}}\left(2\hat{\sigma} + \left(1 - \frac{\mathsf{R}}{\mathsf{r}}\right)\hat{\bar{\lambda}}\right), \tag{5.3.57b}$$

$$\Delta\hat{\rho} = -\mu_{[0]}\hat{\rho} - \hat{\Psi}_2 = \frac{1}{\mathsf{r}}\hat{\rho} - \hat{\Psi}_2. \tag{5.3.57c}$$

Now we have the radial equation for $\hat{\Psi}_0$, being simplified as,

$$\begin{aligned}\Delta\hat{\Psi}_0 &= \delta_{[0]}\hat{\Psi}_1 - \mu_{[0]}\hat{\Psi}_0 - 2\beta_{[0]}\hat{\Psi}_1 + 3\Psi_2^{[0]}\hat{\sigma} \\ &= \eth_{[0]}\hat{\Psi}_1 + \frac{1}{r}\hat{\Psi}_0 - \frac{3R}{2r^3}\hat{\sigma}. \end{aligned} \tag{5.3.58}$$

Further, the corresponding radial equations in Eq. (5.2.13) reduce as,

$$\Delta\hat{\pi} = -\mu_{[0]}\hat{\pi} - \hat{\Psi}_3 = \frac{1}{r}\hat{\pi} - \hat{\Psi}_3, \tag{5.3.59a}$$

$$\Delta\hat{\beta} = -\mu_{[0]}\hat{\beta} - \alpha_{[0]}\hat{\bar{\lambda}} = \frac{1}{r}\hat{\beta} - \frac{\bar{\xi}}{2\sqrt{2}}\hat{\bar{\lambda}}, \tag{5.3.59b}$$

$$\Delta\hat{\alpha} = -\beta_{[0]}\hat{\lambda} - \mu_{[0]}\hat{\alpha} - \hat{\Psi}_3 = \frac{1}{r}\hat{\alpha} + \frac{\xi}{2\sqrt{2}}\hat{\lambda} - \hat{\Psi}_3, \tag{5.3.59c}$$

$$\Delta\hat{\kappa} = -\rho_{[0]}\hat{\bar{\pi}} - \hat{\Psi}_1 = \frac{1}{2r}\left(1 - \frac{R}{r}\right)\hat{\bar{\pi}} - \hat{\Psi}_1, \tag{5.3.59d}$$

$$\Delta\hat{\epsilon} = -\alpha_{[0]}\hat{\bar{\pi}} - \beta_{[0]}\hat{\pi} - \hat{\Psi}_2 = \frac{\xi}{2\sqrt{2}}\hat{\pi} - \frac{\bar{\xi}}{2\sqrt{2}}\hat{\bar{\pi}} - \hat{\Psi}_2. \tag{5.3.59e}$$

5.4 Electromagnetic Background

We extend the case of spherical symmetry to the Reissner-Nordström spacetime, and study the distorted charge distribution through the electromagnetic surficial Love numbers, and the induced multipole moment through the electromagnetic far-field Love numbers, in addition to the surficial and gravitational Love numbers that describe the deformation of the surface, and the induced gravitational multipole moment.

For the charged spherically symmetric background spacetime, we have the unperturbed spin coefficients from Eq. (4.3.6) and Eq. (4.3.12), as,

$$\tau_{[0]} = \nu_{[0]} = \lambda_{[0]} = \sigma_{[0]} = \pi_{[0]} = \kappa_{[0]} = 0, \tag{5.4.1a}$$

$$\mu_{[0]} = -\frac{1}{r}, \tag{5.4.1b}$$

$$\epsilon_{[0]} = \frac{R}{4r^2} - \frac{q^2}{2r^3}\left(1 - \frac{r}{2R}\right), \quad \rho_{[0]} = -\frac{1}{2r}\left(1 - \frac{R}{r} + \frac{q^2}{r^2}\left(1 - \frac{r}{R}\right)\right), \tag{5.4.1c}$$

$$a_{[0]} = \frac{a_{[0]}^{(0)}}{r} = \frac{\bar{\xi}}{\sqrt{2}Rr} \tag{5.4.1d}$$

where we have used the holomorphic coordinates for the connection.

Similarly, the appropriate frame functions and the directional derivatives from Eq. (4.3.13) are,

$$D_{[0]} = \partial_{\mathrm{v}} - \left(1 - \frac{\mathrm{R}}{\mathrm{r}} + \frac{\mathrm{q}^2}{\mathrm{r}^2}\left(1 - \frac{\mathrm{r}}{\mathrm{R}}\right)\right)\partial_{\mathrm{r}}, \tag{5.4.1e}$$

$$\Delta_{[0]} = -\partial_{\mathrm{r}}, \quad \delta_{[0]} = \frac{P_{[0]}}{\mathrm{r}}\partial_{\xi} \tag{5.4.1f}$$

Further, the background Weyl tensors at the horizon from Eq. (4.3.9),

$$\Psi_0^{[0]} = \Psi_1^{[0]} = \Psi_3^{[0]} = \Psi_4^{[0]} = 0, \tag{5.4.1g}$$

$$\Psi_2^{[0]} = -\frac{\mathrm{R}}{2\mathrm{r}^3} + \frac{\mathrm{q}^2}{\mathrm{r}^4}\left(1 - \frac{\mathrm{r}}{2\mathrm{R}}\right), \tag{5.4.1h}$$

and the corresponding unperturbed field tensors from Eq. (4.3.7) are,

$$\phi_0^{[0]} = \phi_2^{[0]} = 0, \quad \left|\phi_1^{[0]}\right|^2 = \frac{\mathrm{q}^2}{4\mathrm{r}^4}. \tag{5.4.1i}$$

5.4.1 Intrinsic Perturbations

We work with the physical choice of $\hat{\phi}_0 \doteq 0$ on the horizon, since we ensure there is no energy flux transferring across the null hypersurface, even in the perturbed spacetime. Further, we continue with the gauge choices used for the spherically symmetric background, specifically $\hat{\tau} = \hat{\nu} = \hat{\gamma} = 0$, motivated from Eq. (4.1.5), $\hat{\epsilon} \doteq \hat{\bar{\epsilon}}$ and $\hat{\epsilon} \doteq 0$ for the area of the perturbed horizon to remain invariant.

The perturbed field equations in Eq. (5.2.16), can be simplified using the unperturbed spin coefficients in Eq. (5.4.1), as,

$$D_{[0]}\hat{\rho} \doteq \hat{\rho}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) \doteq \left(\frac{1}{2\mathrm{R}} - \frac{\mathrm{q}^2}{4\mathrm{R}^2}\right)\hat{\rho}, \tag{5.4.2a}$$

$$D_{[0]}\hat{\sigma} \doteq \hat{\sigma}\left(\epsilon_{[0]} + \bar{\epsilon}_{[0]}\right) \doteq \left(\frac{1}{2\mathrm{R}} - \frac{\mathrm{q}^2}{4\mathrm{R}^2}\right)\hat{\sigma}. \tag{5.4.2b}$$

Similar to the case of the spherically symmetric background, we can solve for both $\hat{\rho}$, and $\hat{\sigma}$ by choosing the perturbation at $\mathrm{v} = 0$ as $\hat{\rho} \doteq \hat{\sigma} \doteq 0$ resembling the unperturbed horizon. Thereby, we have the evolution of the expansion and shear,

$$\hat{\rho}(\mathrm{v}) \doteq \hat{\sigma}(\mathrm{v}) \doteq 0. \tag{5.4.3}$$

Consequently, the perturbed horizon is still characterized by a surface of vanishing expansion and shear.

Additionally, the relations in Eq. (5.2.16) for the perturbations of the transversal expansion ($\hat{\mu} \doteq 0$) and shear simplify using the spin operator with $\hat{\pi} : \mathfrak{s} = -1$ from Eq. (2.3.18),

$$D_{[0]}\hat{\mu} \doteq \eth_{[0]}\hat{\pi} + \hat{\Psi}_2 \doteq 0, \tag{5.4.4a}$$

$$D_{[0]}\hat{\lambda} \doteq -\frac{1}{2\mathsf{R}}\hat{\lambda} + \bar{\eth}_{[0]}\hat{\pi}. \tag{5.4.4b}$$

Hence, we have the differential equation satisfied by $\hat{\pi}$ on the horizon, which can be solved by manifesting $\hat{\Psi}_2$ in terms of the spherical harmonics, and using the properties of the spin operator,

$$\eth_{[0]}\hat{\pi} \doteq -\hat{\Psi}_2. \tag{5.4.5}$$

Further, using the background unperturbed quantities from Eq. (5.4.1) in the Maxwell field equations from Eq. (5.2.15), we have,

$$D_{[0]}\hat{\phi}_1 \doteq 0, \tag{5.4.6a}$$

$$D_{[0]}\hat{\phi}_2 \doteq 2\phi_1^{[0]}\hat{\pi}. \tag{5.4.6b}$$

where we assume spherical symmetry for $\phi_1^{[0]}$ and $\hat{\phi}_1$.

APPENDICES

If one wants to summarize our knowledge of physics in the briefest possible terms, there are three really fundamental observations: (i) Spacetime is a pseudo-Riemannian manifold $\mathscr{M}$, endowed with a metric tensor and governed by geometrical laws. (ii) Over $\mathscr{M}$ is a vector bundle $\boldsymbol{X}$ with a non-abelian gauge group G. (iii) Fermions are sections of $(\hat{S}_{+} \otimes V_{R}) \oplus (\hat{S}_{-} \otimes V_{\bar{R}})$. R and $\bar{R}$ are not isomorphic; their failure to be isomorphic explains why the light fermions are light and presumably has its origins in a representation difference Δ in some underlying theory. All of this must be supplemented with the understanding that the geometrical laws obeyed by the metric tensor, the gauge fields, and the fermions are to be interpreted in quantum mechanical terms.

EDWARD WITTEN

Appendix A

Mathematical Preliminaries

A.1 Differential Forms and Exterior Calculus

We begin by defining p-forms, which are multilinear form fields, completely antisymmetric with respect to all p arguments. We define the *exterior product* $\wedge$, as the 2-form $\boldsymbol{\omega} \wedge \boldsymbol{\sigma}$ defined from the 1-forms $\boldsymbol{\omega}$, and $\boldsymbol{\omega}$, by, $\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = \boldsymbol{\omega} \otimes \boldsymbol{\sigma} - \boldsymbol{\sigma} \otimes \boldsymbol{\omega}$, which manifests the antisymmetry.

Further, the *exterior derivative* is defined sequentially, with df being the 1-form gradient of f for any scalar field f. Extending out, for any $p+q$-form $\boldsymbol{\Omega}$, which can be written as the exterior product of a p-form $\boldsymbol{\alpha}$, and a q-form $\boldsymbol{\beta}$, we have the extended Leibinitz rule, $d\boldsymbol{\Omega} = d(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = d\boldsymbol{\alpha} \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge d\boldsymbol{\beta}$.

The components of the exterior derivative of a 1-form $\boldsymbol{\omega}$, are explicitly $(d\omega)_{\alpha\beta} = \partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha$, where the partial derivative defined on the coordinate system, can be replaced by any covariant derivative operator without torsion on $\mathscr{M}$.

An essential relation relating the Lie derivative of a p-form $\boldsymbol{\omega}$ along a vector field, $\boldsymbol{v}$, to the exterior derivative of $\boldsymbol{\omega}$, as $\mathcal{L}_{\boldsymbol{v}} \boldsymbol{\omega} = \boldsymbol{v} \cdot d\boldsymbol{\omega} + d(\boldsymbol{v} \cdot \boldsymbol{\omega})$, which leads to the Cartan's identity.

A.2 Confluent Heun Equations

We have the canonical form of the Heun linear differential equation, with four regular singular points, transformed as,

$$\partial_z^2 \mathcal{R} + \left(\frac{c}{z} + \frac{d}{z-1} + \frac{e}{z-f} \right) \partial_z \mathcal{R} + \frac{abz - q}{z(z-1)(z-f)} \mathcal{R} = 0, \tag{A.2.1}$$

with the relation between the coefficients, $a+b+1 = c+d+e$. This generalizes the *hypergeometric equation*, which is realized when $f = 1$ and $q = ab$ and factorize.

We obtain the confluent form of the above equation by coalescing two regular singular points,

$$\partial_z^2 \mathcal{R} + \left(\alpha + \frac{\gamma+1}{z-1} + \frac{\beta+1}{z} \right) \partial_z \mathcal{R} + \left(\frac{\nu}{z-1} \right) + \frac{\mu}{z} \mathcal{R} = 0, \tag{A.2.2}$$

with solution,

$$\mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z), \tag{A.2.3}$$

where $\delta = \mu + \nu - \alpha\left(1 + \frac{\beta+\gamma}{2}\right)$, $\eta = \frac{(\alpha-\gamma)(1+\beta)-\beta}{2} - \mu$.

The canonical solution is defined by the convergent Taylor series expansion around the origin, and the normalization,

$$\mathcal{R}(z) = \mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta; 0) = 1 \tag{A.2.4a}$$

$$\partial_z \mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z)\Big|_{z=0} = \frac{1}{2} \tag{A.2.4b}$$

A.3 Hypergeometric functions

Appendix B

Newman-Penrose formalism for Kerr

B.1 Essential Properties

The Kerr metric with mass M and spin a is given by the line element in the Boyer-Lindquist coordinates as

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 \\ &+\rho^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}d\phi^2, \end{aligned} \tag{B.1.1}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta, \\ \Delta &= r^2 - 2Mr + a^2, \\ \Sigma^2 &= (r^2+a^2)\rho^2 + 2a^2Mr\sin^2\theta. \end{aligned} \tag{B.1.2}$$

Thereby, we have the metric as

$$g_{ij} = \begin{pmatrix} -\left(1-\frac{2Mr}{\rho^2}\right) & 0 & 0 & -\frac{2aMr\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2aMr\sin^2\theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2\sin^2\theta}{\rho^2} \end{pmatrix}, \tag{B.1.3a}$$

and the corresponding inverse as

$$g^{ij} = \begin{pmatrix} -\frac{\Sigma^2}{\rho^2\Delta} & 0 & 0 & -\frac{2aMr}{\rho^2\Delta} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ -\frac{2aMr}{\rho^2\Delta} & 0 & 0 & \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta} \end{pmatrix}. \tag{B.1.3b}$$

We have a coordinate singularity at $\Delta = 0$, which happens for $r \equiv r_\pm = M \pm \sqrt{M^2 - a^2}$; thereby, we look for a family of null geodesics and choose a coordinate system such that the null geodesics are coordinate lines in the new system. The Kerr family admits two special families of null geodesics, termed the *principal null geodesics*, given by

$$u^i = \frac{dx^i}{d\lambda} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda}\right) = \left(\frac{r^2+a^2}{\Delta}, \pm 1, 0, \frac{a}{\Delta}\right), \tag{B.1.4}$$

where the sign corresponds to the nature of ingoing or outgoing geodesics. Let us consider the outgoing geodesics, by indicating the tangent vector as

$$\ell^i \equiv \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a), \tag{B.1.5}$$

which is parametrized in terms of r as

$$\frac{dt}{dr} = -\frac{r^2+a^2}{\Delta}, \quad \text{(B.1.6a)} \qquad \frac{d\phi}{dr} = -\frac{a}{\Delta}. \quad \text{(B.1.6b)}$$

We want these geodesics to be coordinate lines of our new system, which mitigates the coordinate singularity. We choose one of the coordinates as r, followed by θ which is constant along the geodesic. The further constant coordinates are designed to be constant along the geodesic family, as

$$v = t + T(r), \quad \text{(B.1.7a)} \qquad \varphi = \phi + \Phi(r), \quad \text{(B.1.7b)}$$

such that, we have $\frac{dv}{dr} = \frac{d\varphi}{dr} = 0$, and the coordinate transformation $(t, r, \theta, \phi) \to (v, r, \theta, \varphi)$ with

$$dv = dt + \frac{r^2+a^2}{\Delta}dr, \quad \text{(B.1.8a)} \qquad d\varphi = d\phi - \frac{a}{\Delta}dr, \quad \text{(B.1.8b)}$$

which yields the following metric

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)dv^2 + 2dvdr - 2a\sin^2\theta drd\varphi \\ &- \frac{4aMr\sin^2\theta}{\rho^2}dvd\varphi + \rho^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}d\varphi^2, \end{aligned} \tag{B.1.9}$$

where the horizon is a 3-dimensional surface $r = r_+$, which is denoted as Δ.

B.2 Newman-Penrose formalism for Kerr Metric

B.2.1 Tetrad formalism

We have the family of null geodesics in Eq. (B.1.4) given by the tangent vectors

$$\frac{dt}{d\tau} = \frac{r^2+a^2}{\Delta}E, \tag{B.2.1a}$$

$$\frac{dr}{d\tau} = \pm E, \tag{B.2.1b}$$
$$\frac{d\theta}{d\tau} = 0, \tag{B.2.1c}$$
$$\frac{d\phi}{d\tau} = \frac{a}{\Delta}E, \tag{B.2.1d}$$

where E is a constant. We define the real null-vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ of the Newman-Penrose formalism in terms of these geodesics and adjoining to them a complex null vector $\boldsymbol{m}$ as

$$\ell^i \equiv \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \tag{B.2.2a}$$
$$n^i \equiv \frac{1}{2\rho^2}(r^2 + a^2, -\Delta, 0, a), \tag{B.2.2b}$$
$$m^i \equiv \frac{1}{\tilde{\rho}\sqrt{2}}(ia\sin\theta, 0, 1, i\csc\theta), \tag{B.2.2c}$$

where we denote

$$\tilde{\rho} = r + ia\cos\theta, \tag{B.2.3}$$

such that $\rho^2 = \tilde{\rho}\bar{\tilde{\rho}} = r^2 + a^2\cos^2\theta$. Note that the basis tetrads satisfy the normalization conditions in Eq. (2.2.3). Further, we have the covectors corresponding to the tetrad as

$$\ell_i = \frac{1}{\Delta}\left(\Delta, -\rho^2, 0, -a\Delta\sin^2\theta\right), \tag{B.2.4a}$$
$$n_i = \frac{1}{2\rho^2}\left(\Delta, \rho^2, 0, -a\Delta\sin^2\theta\right), \tag{B.2.4b}$$
$$m_i = \frac{1}{\tilde{\rho}\sqrt{2}}\left(ia\sin\theta, 0, -\rho^2, -i(r^2 + a^2)\sin\theta\right), \tag{B.2.4c}$$

which we compute by the fundamental relation for the inverse in Eq. (2.1.3).

B.2.2 Spin Coefficients

We compute the basis 1-forms in the Boyer-Lindquist coordinates as

$$\boldsymbol{\ell} \equiv \ell_i dx^i = dt - \frac{\rho^2}{\Delta}dr - a\sin^2\theta d\phi, \tag{B.2.5a}$$
$$\boldsymbol{n} \equiv n_i dx^i = \frac{\Delta}{2\rho^2}dt + \frac{1}{2}dr - \frac{a\Delta\sin^2\theta}{2\rho^2}d\phi, \tag{B.2.5b}$$
$$\boldsymbol{m} \equiv m_i dx^i = \frac{ia\sin\theta}{\tilde{\rho}\sqrt{2}}dt - \frac{\bar{\tilde{\rho}}}{\sqrt{2}}d\theta - \frac{i(r^2 + a^2)\sin\theta}{\tilde{\rho}\sqrt{2}}d\phi, \tag{B.2.5c}$$

with the exterior derivatives for the basis 1-forms being noted through

$$
\begin{aligned}
d\boldsymbol{\ell} &= -d\left(\frac{\rho^2}{\Delta}\right)\wedge dr - d\left(a\sin^2\theta\right)\wedge d\phi \\
&= -\left(\partial_\theta\left(\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2}\right)d\theta\right)\wedge dr - \left(\partial_\theta\left(a\sin^2\theta\right)d\theta\right)\wedge d\phi \\
&= \frac{2a^2\cos\theta\sin\theta}{\Delta}d\theta\wedge dr - 2a\sin\theta\cos\theta\; d\theta\wedge d\phi, \qquad \text{(B.2.6a)}
\end{aligned}
$$

$$
\begin{aligned}
d\boldsymbol{n} &= d\left(\frac{\Delta}{2\rho^2}\right)\wedge dt - d\left(\frac{a\Delta\sin^2\theta}{2\rho^2}\right)\wedge d\phi \\
&= \left(\partial_r\left(\frac{r^2-2Mr+a^2}{2(r^2+a^2\cos^2\theta)}\right)dr + \partial_\theta\left(\frac{r^2-2Mr+a^2}{2(r^2+a^2\cos^2\theta)}\right)d\theta\right)\wedge dt \\
&\quad -\left(\partial_r\left(\frac{a(r^2-2Mr+a^2)\sin^2\theta}{2(r^2+a^2\cos^2\theta)}\right)dr\right. \\
&\quad \left.+\partial_\theta\left(\frac{a(r^2-2Mr+a^2)\sin^2\theta}{2(r^2+a^2\cos^2\theta)}\right)d\theta\right)\wedge d\phi \\
&= \left(\frac{(2r-2M)\rho^2-2r\Delta}{2\rho^4}\right)dr\wedge dt - \frac{2a^2\Delta\cos\theta\sin\theta}{2\rho^4}d\theta\wedge dt \\
&\quad -\left(\frac{(2r-2M)a\rho^2\sin^2\theta-2ar\Delta\sin^2\theta}{2\rho^4}\right)dr\wedge d\phi \\
&\quad -\left(\frac{2a\rho^2\Delta\sin\theta\cos\theta-4a^3\Delta\sin^3\theta\cos\theta}{2\rho^4}\right)d\theta\wedge d\phi, \qquad \text{(B.2.6b)}
\end{aligned}
$$

$$
\begin{aligned}
d\boldsymbol{m} &= d\left(\frac{ia\sin\theta}{\tilde\rho\sqrt2}\right)\wedge dt - d\left(\frac{\bar{\tilde\rho}}{\sqrt2}\right)\wedge d\theta - d\left(\frac{i(r^2+a^2)\sin\theta}{\tilde\rho\sqrt2}\right)\wedge d\phi \\
&= \left(\partial_r\left(\frac{ia\sin\theta}{\sqrt2(r+ia\cos\theta)}\right)dr + \partial_\theta\left(\frac{ia\sin\theta}{\sqrt2(r+ia\cos\theta)}\right)d\theta\right)\wedge dt \\
&\quad -\left(\partial_r\left(\frac{r-ia\cos\theta}{\sqrt2}\right)dr\right)\wedge d\theta - \left(\partial_r\left(\frac{i(r^2+a^2)\sin\theta}{\sqrt2(r+ia\cos\theta)}\right)dr\right. \\
&\quad \left.+\;\partial_\theta\left(\frac{i(r^2+a^2)\sin\theta}{\sqrt2(r+ia\cos\theta)}\right)d\theta\right)\wedge d\phi \\
&= -\frac{ia\sin\theta}{\tilde\rho^2\sqrt2}dr\wedge dt - \left(\frac{a^2-iar\cos\theta}{\tilde\rho^2\sqrt2}\right)d\theta\wedge dt - \frac{1}{\sqrt2}dr\wedge d\theta \\
&\quad -\left(\frac{i\sin\theta\left(2r\tilde\rho-(r^2+a^2)\right)}{\tilde\rho^2\sqrt2}\right)dr\wedge d\phi + \left(\frac{i(r^2+a^2)(ir\cos\theta-a)}{\tilde\rho^2\sqrt2}\right)d\theta\wedge d\phi. \qquad \text{(B.2.6c)}
\end{aligned}
$$

Now we reexpress the null tetrad basis wedges in terms of the coordinate wedges in Eq. (2.2.14) through the expressions in Eq. (B.2.5). Comparing the coefficients to the derivative expressions of the basis 1-forms in Eq. (B.2.6), we have the spin coefficients for the Kerr metric as

$$\kappa = \sigma = \lambda = \nu = \epsilon = 0 \tag{B.2.7a}$$

$$\rho = -\frac{1}{\bar{\tilde{\rho}}}; \quad \beta = \frac{\cot\theta}{2\sqrt{2}\tilde{\rho}}; \quad \pi = \frac{ia\sin\theta}{\bar{\tilde{\rho}}^2\sqrt{2}}; \quad \tau = -\frac{ia\sin\theta}{\rho^2\sqrt{2}} \tag{B.2.7b}$$

$$\mu = -\frac{\Delta}{2\rho^2\bar{\tilde{\rho}}}; \quad \gamma = \mu + \frac{r-M}{2\rho^2}; \quad \alpha = \pi - \bar{\beta} \tag{B.2.7c}$$

As a result, we have analogous equations that provide a more comprehensive understanding of the physical intricacies involved.

$$D\boldsymbol{\ell} = 0 \tag{B.2.8a}$$

$$D\boldsymbol{n} = \frac{ia\sin\theta}{\bar{\tilde{\rho}}^2}\boldsymbol{m} - \frac{ia\sin\theta}{\tilde{\rho}^2}\bar{\boldsymbol{m}} \tag{B.2.8b}$$

$$D\boldsymbol{m} = -\frac{ia\sin\theta}{\tilde{\rho}^2}\boldsymbol{\ell} \tag{B.2.8c}$$

$$\Delta\boldsymbol{\ell} = \left(-\frac{\Delta}{2\rho^2}\left(\frac{1}{\tilde{\rho}} + \frac{1}{\bar{\tilde{\rho}}}\right) + \frac{r-M}{\rho^2}\right)\boldsymbol{\ell} - \frac{ia\sin\theta}{\sqrt{2}\rho^2}\boldsymbol{m} + \frac{ia\sin\theta}{\sqrt{2}\rho^2}\bar{\boldsymbol{m}} \tag{B.2.8d}$$

$$\Delta\boldsymbol{n} = -\left(-\frac{\Delta}{2\rho^2}\left(\frac{1}{\tilde{\rho}} + \frac{1}{\bar{\tilde{\rho}}}\right) + \frac{r-M}{\rho^2}\right)\boldsymbol{n} \tag{B.2.8e}$$

$$\Delta\boldsymbol{m} = \frac{ia\sin\theta}{\sqrt{2}\rho^2}\boldsymbol{n} + \frac{\Delta}{2\rho^2}\left(\frac{1}{\tilde{\rho}} - \frac{1}{\bar{\tilde{\rho}}}\right)\boldsymbol{m} \tag{B.2.8f}$$

$$\delta\boldsymbol{\ell} = -\frac{ia\sin\theta}{\sqrt{2}\tilde{\rho}^2}\boldsymbol{\ell} + \frac{1}{\tilde{\rho}}\boldsymbol{m} \tag{B.2.8g}$$

$$\delta\boldsymbol{n} = \frac{ia\sin\theta}{\sqrt{2}\tilde{\rho}^2}\boldsymbol{n} - \frac{\Delta}{2\rho^2\bar{\tilde{\rho}}}\boldsymbol{m} \tag{B.2.8h}$$

$$\delta\boldsymbol{m} = \left(\frac{\cot\theta}{\sqrt{2}\tilde{\rho}} + \frac{ia\sin\theta}{\sqrt{2}\tilde{\rho}^2}\right)\boldsymbol{m} \tag{B.2.8i}$$

$$\bar{\delta}\boldsymbol{m} = -\frac{\Delta}{2\rho^2\tilde{\rho}}\boldsymbol{\ell} + \frac{1}{\bar{\tilde{\rho}}}\boldsymbol{n} + \left(\frac{ia\sin\theta}{\sqrt{2}\bar{\tilde{\rho}}^2} - \frac{\cot\theta}{\sqrt{2}\bar{\tilde{\rho}}}\right)\boldsymbol{m} \tag{B.2.8j}$$

Thus, $\boldsymbol{\ell}$ is tangential to null geodesics and is further affinely parameterized. Similarly, $\boldsymbol{n}$ is geodesic but non-affinely parameterized with acceleration given by the spin coefficients $\gamma + \bar{\gamma}$.

B.2.3 Weyl Scalars

We begin with the set of Newman-Penrose field equations from Eq. (2.2.26) in vacuum, with the Ricci scalars vanishing, thus leading to,

$$D\sigma - \delta\kappa = \sigma(3\epsilon - \bar{\epsilon} + \rho + \bar{\rho}) + \kappa(\bar{\pi} - \tau - 3\beta - \alpha) + \Psi_0, \tag{B.2.9a}$$

$$D\rho - \bar{\delta}\kappa = \left(\rho^2 + |\sigma|^2\right) + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau + \kappa\left(\pi - 3\alpha - \bar{\beta}\right), \tag{B.2.9b}$$
$$D\tau - \Delta\kappa = \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) + \Psi_1, \tag{B.2.9c}$$
$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho), \tag{B.2.9d}$$
$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) + \epsilon(\bar{\pi} - \bar{\alpha}) + \Psi_1, \tag{B.2.9e}$$
$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \beta) - \nu\bar{\kappa} + \lambda(\bar{\epsilon} - 3\epsilon), \tag{B.2.9f}$$
$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) - \nu(3\epsilon + \bar{\epsilon}) + \Psi_3, \tag{B.2.9g}$$
$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma\left(\bar{\beta} - \bar{\tau}\right) - \Psi_3, \tag{B.2.9h}$$
$$\Delta\lambda - \bar{\delta}\nu = \lambda(\bar{\gamma} - 3\gamma - \mu - \bar{\mu}) + \nu\left(3\alpha + \bar{\beta} + \pi - \bar{\tau}\right) - \Psi_4, \tag{B.2.9i}$$
$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) + \sigma\left(\bar{\beta} - 3\alpha\right) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1, \tag{B.2.9j}$$
$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu\left(\alpha + \bar{\beta}\right) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3, \tag{B.2.9k}$$
$$\delta\nu - \Delta\mu = \left(\mu^2 + \lambda\bar{\lambda}\right) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}), \tag{B.2.9l}$$
$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} + \beta(\mu - \gamma + \bar{\gamma}) + \alpha\bar{\lambda}, \tag{B.2.9m}$$
$$\delta\tau - \Delta\sigma = \left(\mu\sigma + \bar{\lambda}\rho\right) + \tau(\tau + \beta - \bar{\alpha}) + \sigma(\bar{\gamma} - 3\gamma) - \kappa\bar{\nu}, \tag{B.2.9n}$$
$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \Psi_2, \tag{B.2.9o}$$
$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + \tau\left(\bar{\beta} - \alpha - \bar{\tau}\right) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2, \tag{B.2.9p}$$
$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - 2\gamma\epsilon - \gamma\bar{\epsilon} - \epsilon\bar{\gamma} + \tau\pi - \nu\kappa + \Psi_2, \tag{B.2.9q}$$
$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2. \tag{B.2.9r}$$

We further substituting the known zero spin coefficients, and substituting $\pi = \alpha + \bar{\beta}$, such that it simplifies dramatically to give

$$0 = \Psi_0, \tag{B.2.10a}$$
$$D\rho = \rho^2, \tag{B.2.10b}$$
$$D\tau = \rho(\tau + \bar{\pi}) + \Psi_1, \tag{B.2.10c}$$
$$D\alpha = (\alpha + \pi)\rho, \tag{B.2.10d}$$
$$D\beta = \beta\bar{\rho} + \Psi_1, \tag{B.2.10e}$$
$$-\bar{\delta}\pi = \pi(\pi + \alpha - \beta), \tag{B.2.10f}$$
$$-\Delta\pi = \mu(\pi + \bar{\tau}) + \pi(\gamma - \bar{\gamma}) + \Psi_3, \tag{B.2.10g}$$
$$\Delta\alpha - \bar{\delta}\gamma = \alpha(\bar{\gamma} - \bar{\mu}) + \gamma\left(\bar{\beta} - \bar{\tau}\right) - \Psi_3, \tag{B.2.10h}$$
$$0 = -\Psi_4, \tag{B.2.10i}$$
$$\delta\rho = \rho\bar{\pi} + \tau(\rho - \bar{\rho}) - \Psi_1, \tag{B.2.10j}$$
$$-\bar{\delta}\mu = \pi(\mu - \bar{\mu}) + \mu\pi - \Psi_3, \tag{B.2.10k}$$
$$-\Delta\mu = \mu(\mu + \gamma + \bar{\gamma}), \tag{B.2.10l}$$
$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau + \beta(\mu - \gamma + \bar{\gamma}), \tag{B.2.10m}$$
$$\delta\tau = \tau(\tau + \beta - \bar{\alpha}), \tag{B.2.10n}$$

$$D\mu - \delta\pi = \bar{\rho}\mu + 2\beta\pi + \Psi_2, \tag{B.2.10o}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} + \tau\left(\bar{\beta} - \alpha - \bar{\tau}\right) + \rho(\gamma + \bar{\gamma}) - \Psi_2, \tag{B.2.10p}$$

$$D\gamma = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) + \tau\pi + \Psi_2, \tag{B.2.10q}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) - \Psi_2. \tag{B.2.10r}$$

Directly, from Eq. (B.2.10a) and Eq. (B.2.10i), we have $\Psi_0 = \Psi_4 = 0$. Further, we compute Ψ_1 through Eq. (B.2.10e), wherein

$$\begin{aligned} D\beta = \ell^i \nabla_i \beta &= \partial_t \beta + \partial_r \beta \\ &= \partial_r \left(\frac{\cot\theta}{2\sqrt{2}(r + ia\cos\theta)} \right) = -\frac{\cot\theta}{2\sqrt{2}(r + ia\cos\theta)^2} \\ &= \beta\bar{\rho}, \end{aligned}$$

hence $\Psi_1 = 0$. Similarly, Ψ_3 can be easily computed from Eq. (B.2.10g), where

$$\begin{aligned} \Delta\pi = n^i \nabla_i \pi &= \frac{1}{2\rho^2}\left((r^2 + a^2)\partial_t - \Delta\partial_r + a\partial_\phi\right)\pi \\ &= \frac{-\Delta}{2\rho^2}\partial_r \left(\frac{ia\sin\theta}{(r - ia\cos\theta)^2\sqrt{2}} \right) \\ &= 2\mu\pi, \end{aligned}$$

such that $\Psi_3 = \mu(\pi + \bar{\tau}) - \Delta\pi = 0$ since $\gamma = \bar{\gamma}$ and $\pi = \bar{\tau}$.

Finally, we derive the only non-vanishing Weyl scalar, Ψ_2, using Eq. (B.2.10q), which can be further computed through noting

$$\begin{aligned} D\gamma = \ell^i \nabla_i \gamma &= \partial_t \gamma + \partial_r \gamma \\ &= \partial_r \left(\frac{-(r^2 - 2Mr + a^2)}{2(r^2 + a^2\cos^2\theta)(r - ia\cos\theta)} + \frac{r - M}{2(r^2 + a^2\cos^2\theta)} \right) \\ &= \frac{M}{\tilde{\rho}^3} + 3|\pi|^2 - 2\left(\bar{\pi}\bar{\beta} - \pi\beta\right), \end{aligned}$$

and simplifying as

$$\begin{aligned} \Psi_2 &= D\gamma - 2\alpha\bar{\pi} - 2\beta\pi - |\pi|^2 \\ &= D\gamma + 2\left(\bar{\pi}\bar{\beta} - \pi\beta\right) - 3|\pi|^2, \end{aligned}$$

since $\pi = \bar{\tau}$ and $\alpha = \pi - \bar{\beta}$, thus resulting in, $\Psi_2 = -\frac{M}{(r + ia\cos\theta)^3} = -\frac{M}{\tilde{\rho}^3}$.

Thereby, we have the Weyl scalars of the Kerr metric in the Newman-Penrose formalism as

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = -\frac{M}{\tilde{\rho}^3}. \tag{B.2.11}$$

B.2.4 Regular Coordinates

We now proceed to study the geometry of the 2-surface at the horizon, denoted as Δ, where it is convenient to use regular coordinates, (v, r, θ, φ) such that we employ the metric in Eq. (B.1.9), as

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right) dv^2 + 2dvdr - 2a\sin^2\theta dr d\varphi \\ &- \frac{4aMr\sin^2\theta}{\rho^2} dv d\varphi + \rho^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2} d\varphi^2, \end{aligned} \tag{B.2.12}$$

where the horizon is a 3-dimensional surface $r = r_+$.

We chose the following basis null vectors $\{\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$,

$$\ell^i \equiv \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \tag{B.2.13a}$$

$$n^i \equiv \frac{1}{2\rho^2}(r^2 + a^2, -\Delta, 0, a), \tag{B.2.13b}$$

$$m^i \equiv \frac{1}{\tilde{\rho}\sqrt{2}}(ia\sin\theta, 0, 1, i\csc\theta), \tag{B.2.13c}$$

Note that the basis tetrads satisfy the normalization conditions in Eq. (2.2.3). Further, we have the covectors corresponding to the tetrad as

$$\ell_i = \frac{1}{\Delta}\left(\Delta, -\rho^2, 0, -a\Delta\sin^2\theta\right), \tag{B.2.14a}$$

$$n_i = \frac{1}{2\rho^2}\left(\Delta, \rho^2, 0, -a\Delta\sin^2\theta\right), \tag{B.2.14b}$$

$$m_i = \frac{1}{\tilde{\rho}\sqrt{2}}\left(ia\sin\theta, 0, -\rho^2, -i(r^2 + a^2)\sin\theta\right), \tag{B.2.14c}$$

which we compute by the fundamental relation for the inverse in Eq. (2.1.3).

The basis 1-forms are

$$\boldsymbol{\ell} \equiv \ell_i dx^i = dt - \frac{\rho^2}{\Delta} dr - a\sin^2\theta d\phi, \tag{B.2.15a}$$

$$\boldsymbol{n} \equiv n_i dx^i = \frac{\Delta}{2\rho^2} dt + \frac{1}{2} dr - \frac{a\Delta\sin^2\theta}{2\rho^2} d\phi, \tag{B.2.15b}$$

$$\boldsymbol{m} \equiv m_i dx^i = \frac{ia\sin\theta}{\tilde{\rho}\sqrt{2}} dt - \frac{\bar{\tilde{\rho}}}{\sqrt{2}} d\theta - \frac{i(r^2 + a^2)\sin\theta}{\tilde{\rho}\sqrt{2}} d\phi, \tag{B.2.15c}$$

We have the spin coefficients for the ingoing Kerr metric transformed as

$$\kappa = \sigma = \lambda = \nu = \epsilon = 0 \tag{B.2.16a}$$

$$\rho = -\frac{1}{\bar{\tilde{\rho}}}; \quad \beta = \frac{\cot\theta}{2\sqrt{2}\tilde{\rho}}; \quad \pi = \frac{ia\sin\theta}{\bar{\tilde{\rho}}^2\sqrt{2}}; \quad \tau = -\frac{ia\sin\theta}{\rho^2\sqrt{2}} \tag{B.2.16b}$$

$$\mu = -\frac{\Delta}{2\rho^2 \bar{\tilde{\rho}}}; \quad \gamma = \mu + \frac{r-M}{2\rho^2}; \quad \alpha = \pi - \bar{\beta} \tag{B.2.16c}$$

Further, we have the Weyl scalars of the Kerr metric in the Newman-Penrose formalism as

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 = -\frac{M}{\tilde{\rho}^3}. \tag{B.2.17}$$

app

B.3 Kerr Holomorphic Coordinates

For the null hypersurface being the horizon, we can better understand the geometry by projecting onto holomorphic coordinates $(\xi, \bar{\xi})$. For the ingoing coordinates (v, r, θ, φ) in Eq. (B.2.12), we have the horizon at $r = r_+$, where the intrinsic metric line element is the pull-back of the spacetime metric in Eq. (2.4.15), such that,

$$ds^2 \doteq \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2\theta}{\rho^2} d\varphi^2, \tag{B.3.1}$$

where we are primarily interested in the cross-section of the null hypersurface (horizon) at $r = r_+$. The volume form is given by the determinant of the induced metric,

$$\begin{aligned} \varepsilon &\doteq \Sigma(r_+)\sin\theta \; d\theta \wedge d\varphi \\ &\doteq (r_+^2 + a^2)\sin\theta \; d\theta \wedge d\varphi, \end{aligned} \tag{B.3.2}$$

such that the area of the horizon simplifies as $\mathsf{A} \doteq 4\pi(r_+^2 + a^2)$, with area radius $\mathsf{R} \doteq \sqrt{r_+^2 + a^2}$.

For the metric on the cross section in Eq. (B.3.1), we now change coordinates to $\mu = \cos\theta$, such that, we have

$$ds^2 \doteq \mathsf{R}^2 \left(\frac{d\mu^2}{f(\mu)} + f(\mu) d\varphi^2 \right), \tag{B.3.3a}$$

where we elegantly simplify as,

$$f(\mu) \doteq \frac{1-\mu^2}{1-\beta^2(1-\mu^2)}, \tag{B.3.3b}$$

with $\beta = \frac{a}{\sqrt{2Mr_+}} = \frac{a}{\sqrt{r_+^2 + a^2}}$.

We can now switch to conformal coordinates as in Eq. (2.4.15), with the transformation

$$\xi \doteq \exp(i\varphi)\exp\left(-\beta^2\mu\right)\sqrt{\frac{1+\mu}{1-\mu}}, \tag{B.3.4a}$$

which results in the metric transformation from Eq. (B.3.3a), to arrive at the conformally flat form,

$$ds^2 \doteq \mathsf{R}^2 \frac{f(\mu)}{\xi\bar{\xi}} d\xi d\bar{\xi}, \tag{B.3.4b}$$

such that we have the unit conformal factor, $P(\xi,\bar{\xi}) = \sqrt{\frac{2\xi\bar{\xi}}{f(\mu)}}$. The relationship in Eq. (B.3.4a) can be inverted to have $\mu(\xi,\bar{\xi})$ in the small spin limit. Thus, we have the expansion

$$\frac{f(\xi,\bar{\xi})}{2\xi\bar{\xi}} = \frac{2}{\left(1+\xi\bar{\xi}\right)^2}\left[1+\frac{3a^2}{4M}\frac{1-\xi\bar{\xi}}{\left(1+\xi\bar{\xi}\right)^2}\right]+\mathcal{O}(a^4). \tag{B.3.4c}$$

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2.1 Penrose Representation of the Petrov classification. The Weyl scalars that cannot be eliminated by adjusting the tetrad in the direction of the principal null direction are shown below the corresponding spacetime type, relating to the number of degenerate roots of Eq. (2.3.12). The arrows indicate a progression towards more specialized solutions. 39

2.2 A diagram of a null basis at a point in the spacetime manifold. We have the construction of the null basis $\{\boldsymbol{\ell},\boldsymbol{n},\boldsymbol{m},\bar{\boldsymbol{m}}\}$ adapted to the cross section $\mathscr{N}$ of the null hypersurface $\mathscr{H}$. The orthogonal complement of Span$(\boldsymbol{\ell},\boldsymbol{n})$ is the transverse 2-surface, which is the $(\boldsymbol{m},\bar{\boldsymbol{m}})$ plane. 43

4.1 Representation of the characteristic initial value problem for an Isolated Horizon segment in a vacuum spacetime with adapted null coordinates following the Newman-Penrose formalism. The spacetime is the solution to a characteristic initial value problem with the initial data given on the horizon $\mathscr{H}$, and any null hypersurface, say $\mathscr{H}_0$, intersecting the horizon, with a spacelike cross section $\mathscr{N}_0$. The gray parameters indicate the inclusion of electromagnetic fields. 65

4.2 The characteristic initial value problem constructing the near-horizon geometry of a spherically symmetric spacetime is presented here. The null surface $\mathscr{H}_0$ represents the horizon whose intrinsic geometry is studied, which is foliated by the temporal coordinate v, defined by the tangent vector $\boldsymbol{\ell}$. The null surface $\mathscr{H}$ is the past null cone, generated by past-directed null geodesics originating from a cross-section $\mathscr{N}_0$, defined using holomorphic coordinates $(\xi, \bar{\xi})$. The affine parameter along the geodesics is r, and the null vector is defined through the directional derivative $\Delta = -\partial_{\mathsf{r}}$, whose foliations are presented as dashed cuts. The spacetime metric is constructed starting with suitable data on $\mathscr{H}_0$, $\mathscr{H}$, and $\mathscr{N}_0$. Additionally, the initial data for the projections of the electromagnetic fields pertaining to the Reissner–Nordström metric is presented. 85

5.1 Representation of the region of the spacetime where the expansions of the potential of the tidal deformability are valid. We have an isolated and spherical Newtonian body of mass M and equilibrium radius R. $\mathcal{R}$ denotes the typical length scale of variation of the tidal environment. The form of the total gravitational potential is valid in the region $\mathsf{R} \leq \mathsf{r} \leq \mathcal{R}$. 95

List of Tables

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